Submodular Optimization

CS547 Machine Learning for Big Data

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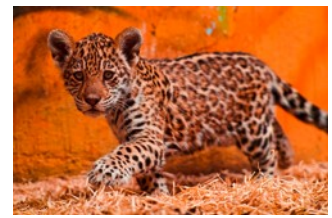
Motivation

- Learned about: LSH/Similarity search & recommender systems
- **Search:** "jaguar"



JAGUAR

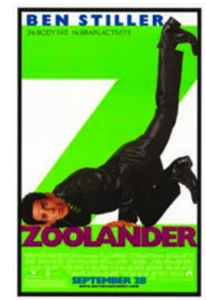
Google

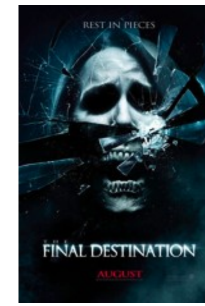
- **Uncertainty** about the user's information need
  - Don't put all eggs in one basket!
- **Relevance** isn't everything – need **diversity**!

Many applications need diversity!

- **Recommendation:**

NETFLIX

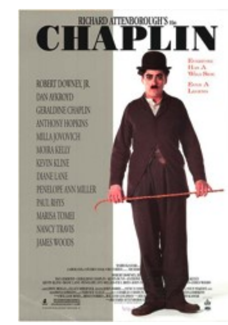




- **Summarization:**

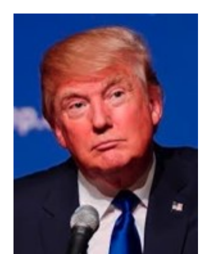
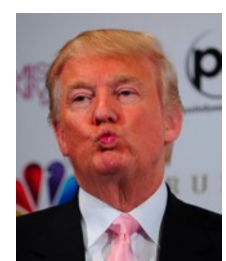
“Robert Downey Jr.”

WIKIPEDIA

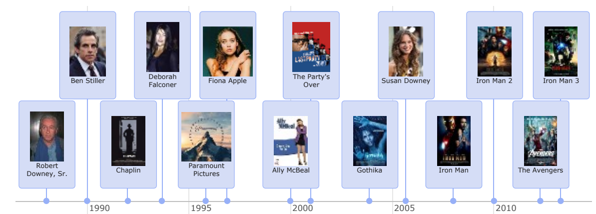


- **News Media:**

YAHOO! NEWS


Automatic Timeline Generation



Person

Timeline

- **Goal:** Timeline should express their **relationships** to other people through **events** (personal, collaboration, mentorship, etc.)

- **Why timelines?**
  - Easier: Wikipedia article is 18 pages long
  - Context: Through relationships & event descriptions
  - Exploration: Can "jump" to other people

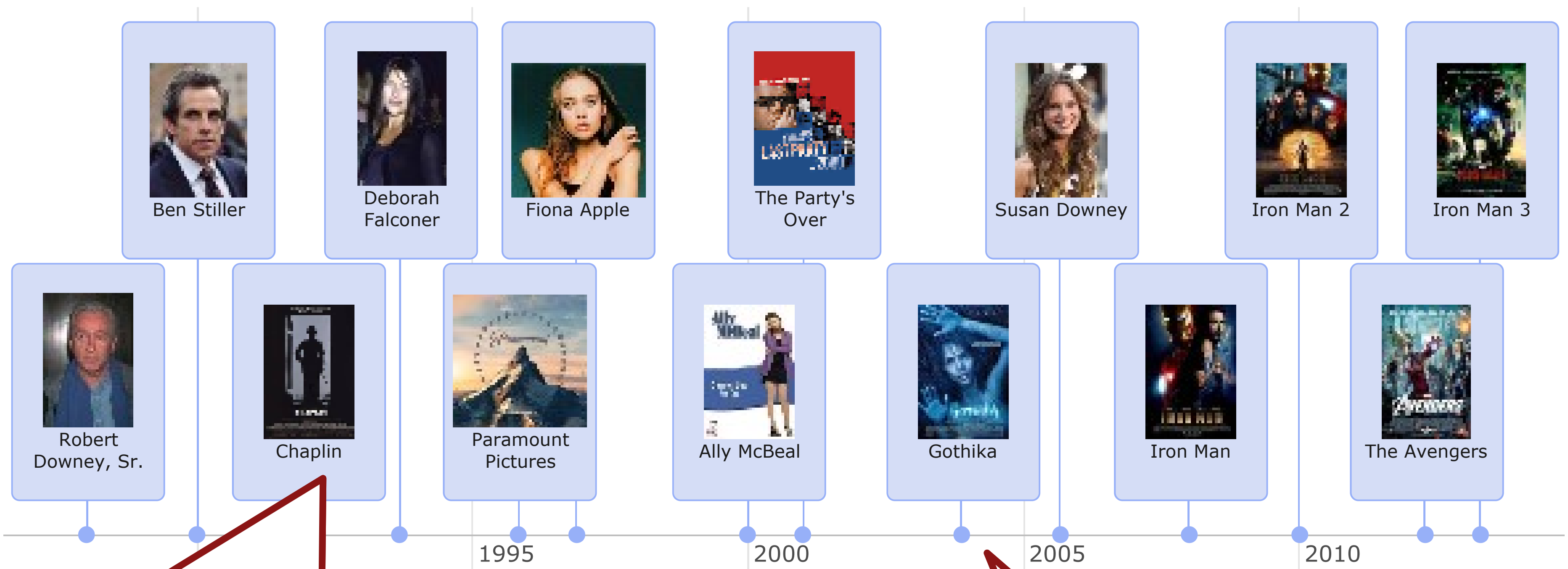
Problem Definition

- **Given:**
  - Relevant **relationships**
  - **Events** that each cover some relationships

- **Goal:** Given a large set of **events**, pick a small subset that explains most known **relationships** ("the timeline")

Demo available at: http://cs.stanford.edu/~althoff/timemachine/demo.html

Example Timeline



"RDJr starred in Chaplin in 1992 together with Anthony Hopkins."

Good overview

Why diversity?

- User studies: People hate redundancy!

Iron Man
US Release

Iron Man
Award
Ceremony

Iron Man
EU Release

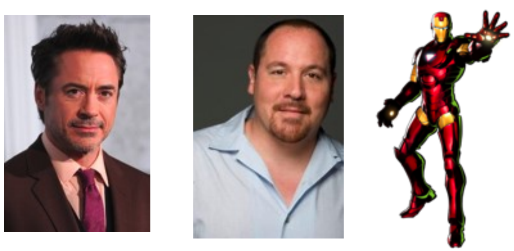
vs

Iron Man
US Release

Chaplin
Academy
Award N.

Rented Lips
US Release

- Want to see more diverse set of relationships



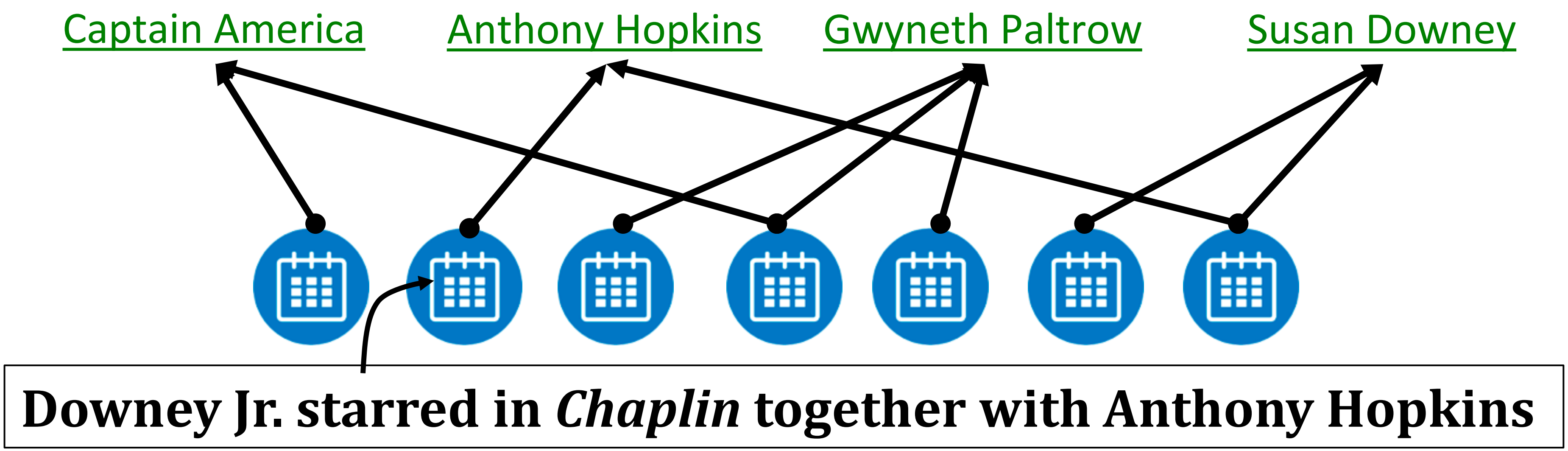
Diversity as Coverage

Encode Diversity as Coverage

- **Idea:** Encode diversity as coverage problem
- **Example:** Selecting events for timeline
  - Try to cover all important relationships

What is being covered?

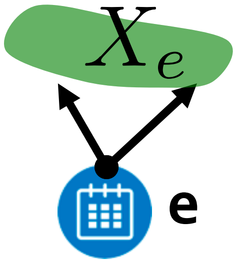
- **Q:** What is being covered?
- **A:** Relationships



- **Q:** Who is doing the covering?
- **A:** Timeline Events

Simple Coverage Model

- Suppose we are given a set of events E
  - Each event **e** covers a set $X_e \subseteq U$ of relationships



- For a set of events $S \subseteq E$ we define:

$$F(S) = \left| \bigcup_{e \in S} X_e \right|$$

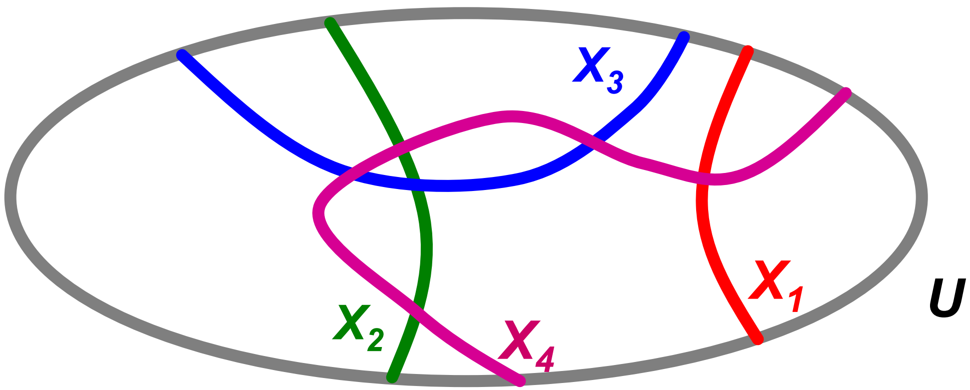
- Goal: We want to $\max_{|S| \leq k} F(S)$

Cardinality Constraint

- Note: F(S) is a set function: $F(S) : 2^E \to \mathbb{N}$

Maximum Coverage Problem

- **Given universe of elements** $U = \{u_1, \ldots, u_n\}$ **and sets** $\{X_1, \ldots, X_m\} \subseteq U$



U: all relationships
X_i: relationships covered by event i

- **Goal: Find set of k events $X_1 \ldots X_k$ covering most of U**
  - **More precisely: Find set of k events $X_1 \ldots X_k$ whose size of the union is the largest**

Simple Greedy Heuristic

Simple Heuristic: Greedy Algorithm:

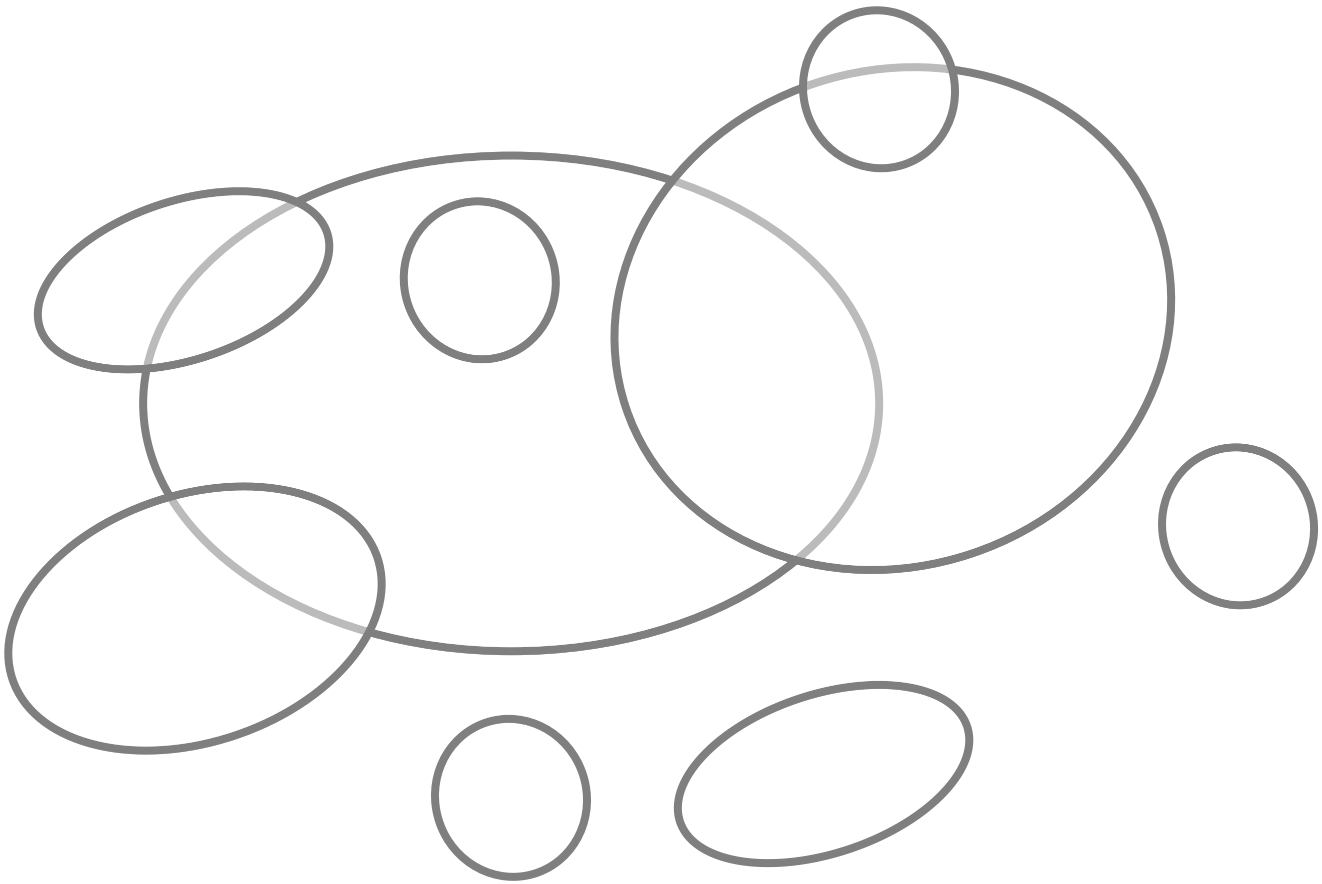
- Start with $S_0 = \{\}$
- For i = 1…k
  - Take event **e** that max $F(S_{i-1} \cup e)$
  - Let $S_i = S_{i-1} \cup \{e\}$

$$F(S) = \left|\bigcup_{e \in S} X_e\right|$$

- **Example:**
  - Eval. **F({e_1}), …, F({e_m})**, pick best (say **e_1**)
  - Eval. **F({e_1} u {e_2}), …, F({e_1} u {e_m})**, pick best (say **e_2**)
  - Eval. **F({e_1, e_2} u {e_3}), …, F({e_1, e_2} u {e_m})**, pick best
  - And so on…

Simple Greedy Heuristic

- **Goal: Maximize the covered area**



Simple Greedy Heuristic

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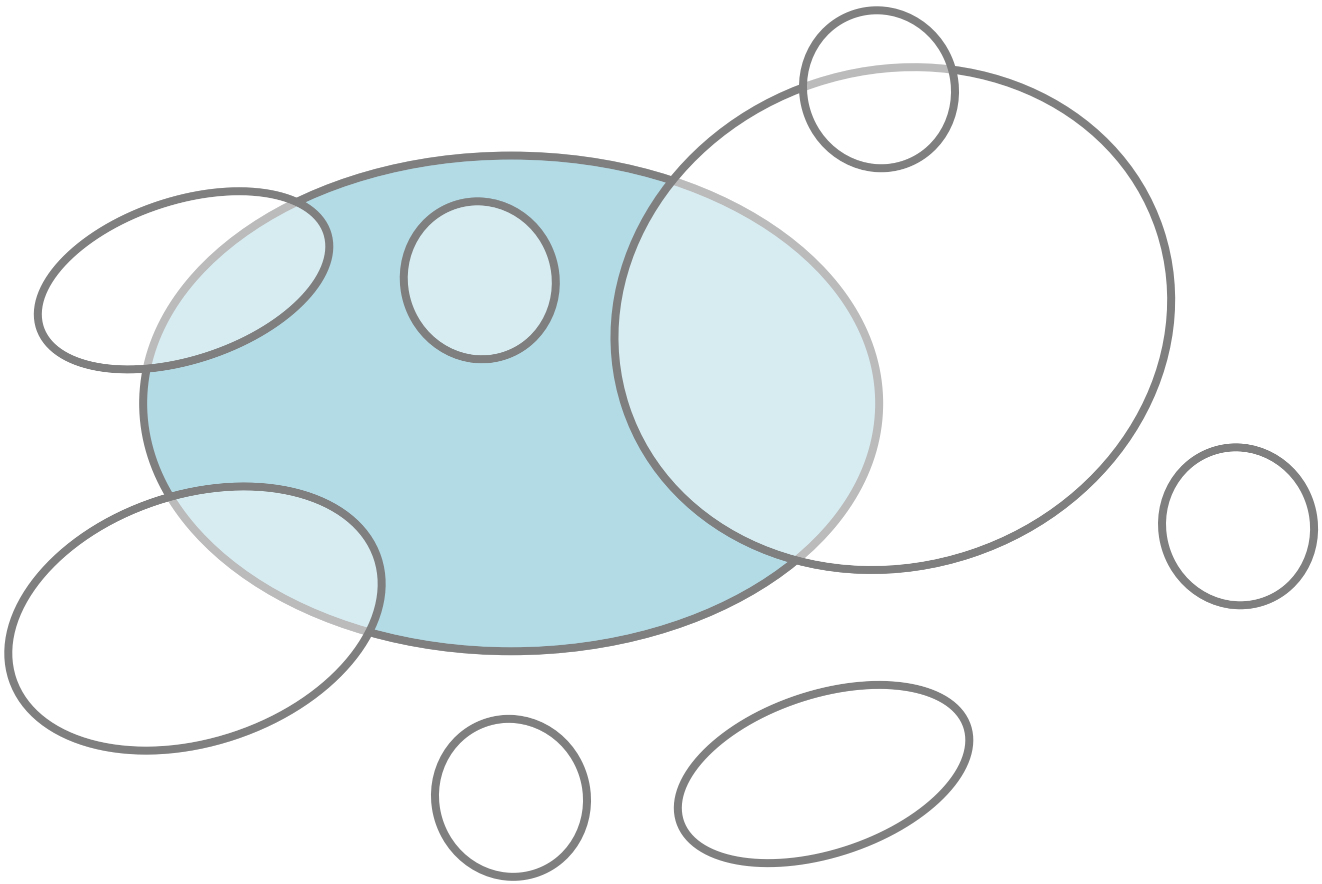

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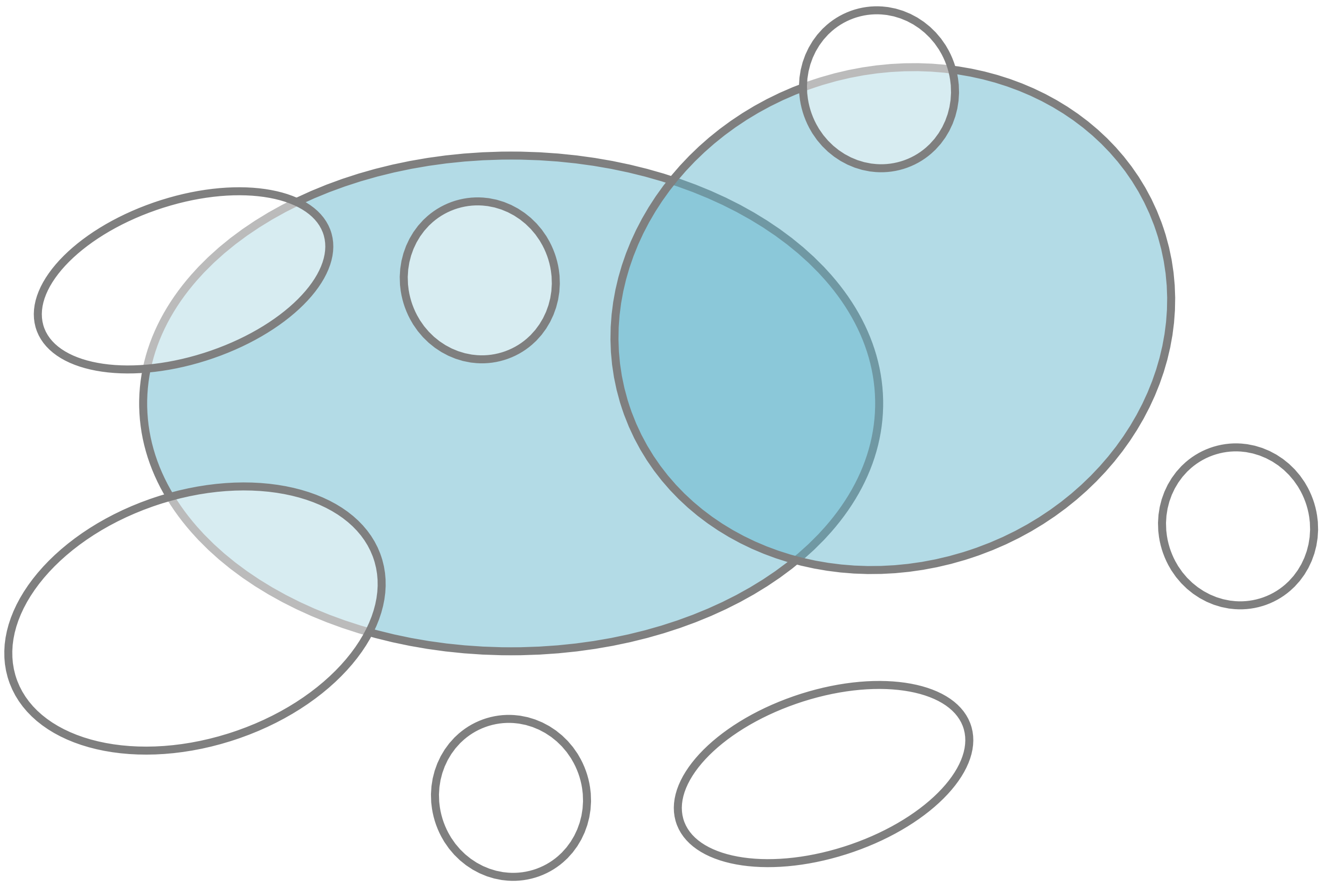

Simple Greedy Heuristic

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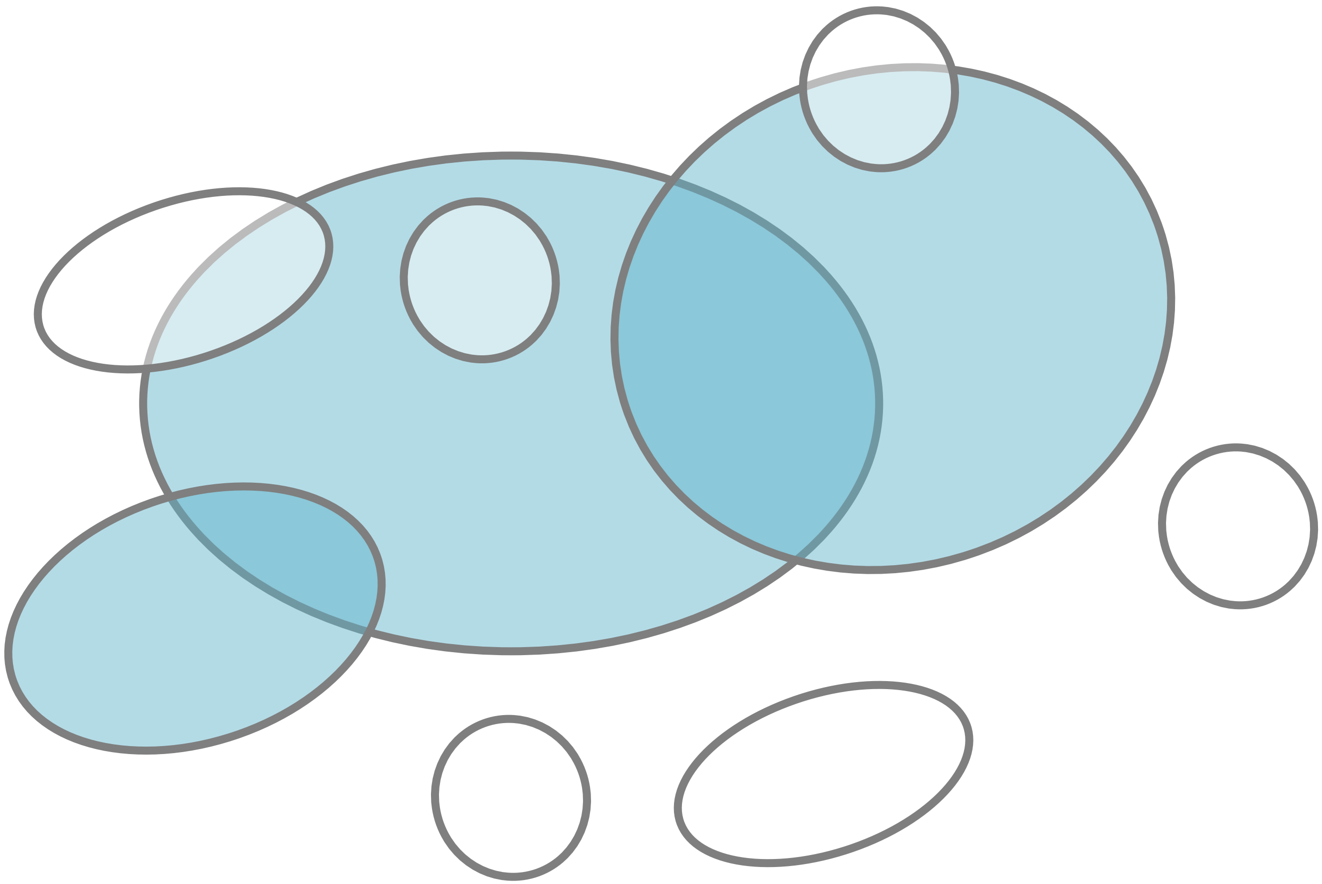
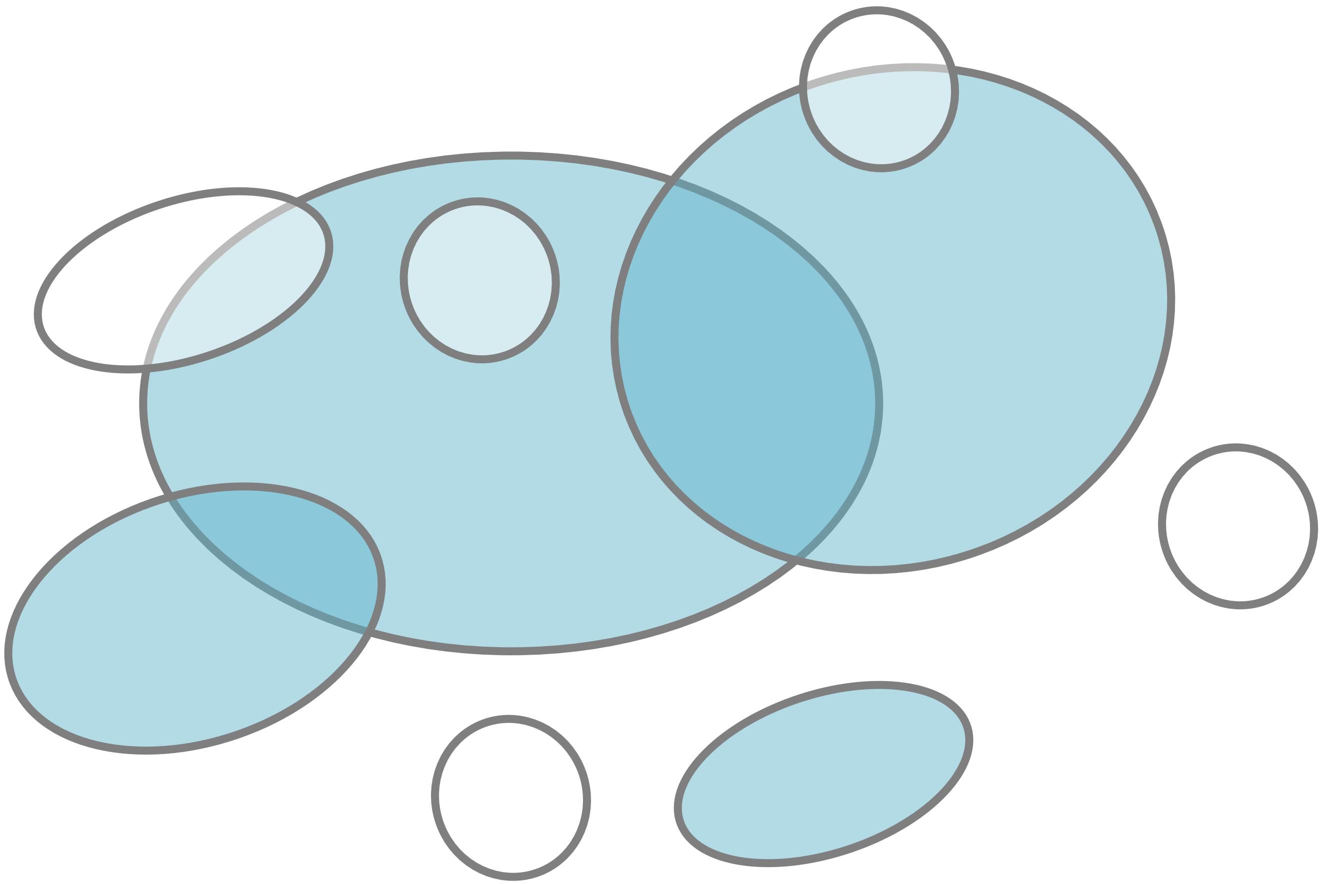
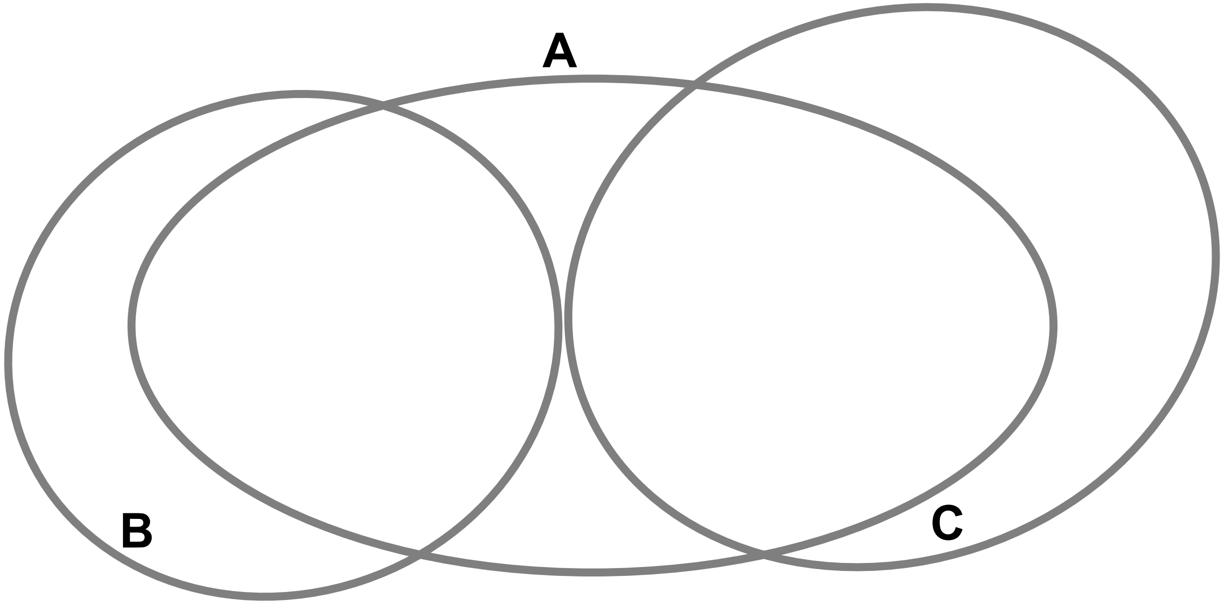


Simple Greedy Heuristic

- **Goal: Maximize the covered area**



When Greedy Heuristic Fails?



- **Goal:** **Maximize the size of the covered area with two sets**
- **Greedy first picks A and then C**
- **But the optimal way would be to pick B and C**

Bad News & Good News

- **Bad news:** Maximum Coverage is NP-hard
  - Related to Set Cover Problem
- **Good news:** Good approximations exist
  - Problem has certain structure to it that even simple greedy algorithms perform reasonably well
  - Details in 2nd half of lecture
- **Now:** **Generalize** our objective for timeline generation

Issue 1: Not all relationships are created equal

- Objective values all relationships **equally**

$$F(S) = \left| \bigcup_{e \in S} X_e \right| = \sum_{r \in R} 1 \text{ where } R = \bigcup_{e \in S} X_e$$

- **Unrealistic:** Some relationships are more important than others
  - use **different weights** ("weighted coverage function")

$$F(S) = \sum_{r \in R} w(r) \qquad\qquad w : R \to \mathbb{R}^+$$

Example weight function

- Use **global importance** weights
- How much interest is there?
- Could be measured as
  - w(X) = # **search queries** for person X
  - w(X) = # **Wikipedia article views** for X
  - w(X) = # **news article mentions** for X

Captain America   Anthony Hopkins   Gwyneth Paltrow   Susan Downey

↓

Captain America Anthony Hopkins Gwyneth Paltrow Susan Downey

Better weight function

Captain America   Justin Bieber   Susan Downey   Tim Althoff

↓ **Applying global importance weights**

$\boxed{\text{Captain America}}$ $\boxed{\text{Justin Bieber}}$ $\boxed{\text{Susan Downey}}$ $\boxed{\text{Tim Althoff}}$

- Some relationships are **not (very) globally important but (not) highly relevant** to timeline
- Need **relevant to timeline** instead of **globally relevant**

w(Susan Downey | RDJr) > w(Justin Bieber | RDJr)

Capturing relevance to timeline

- Can use co-occurrence statistics
  w(X | RDJr) = #(X and RDJr) / (#(RDJr) * #(X))
  - Similar: Pointwise mutual information (PMI)
  - How often do X and Y occur together compared to what you would expect if they were independent
  - Accounts for popular entities (e.g., Justin Bieber)

Issue 2: Differentiating between events

- How to differentiate between two events that **cover the same relationships**?

- **Example:** Robert and Susan Downey
  - **Event 1:** Wedding, August 27, 2005
  - **Event 2:** Minor charity event, Nov 11, 2006

- We need to be able to distinguish these!

Scoring of event timestamps

- Further improvement when we not only score relationships but also **score the event timestamp**

$$F(S) = \sum_{r \in R} w_R(r) + \boxed{\sum_{e \in S} w_T(t_e)}$$

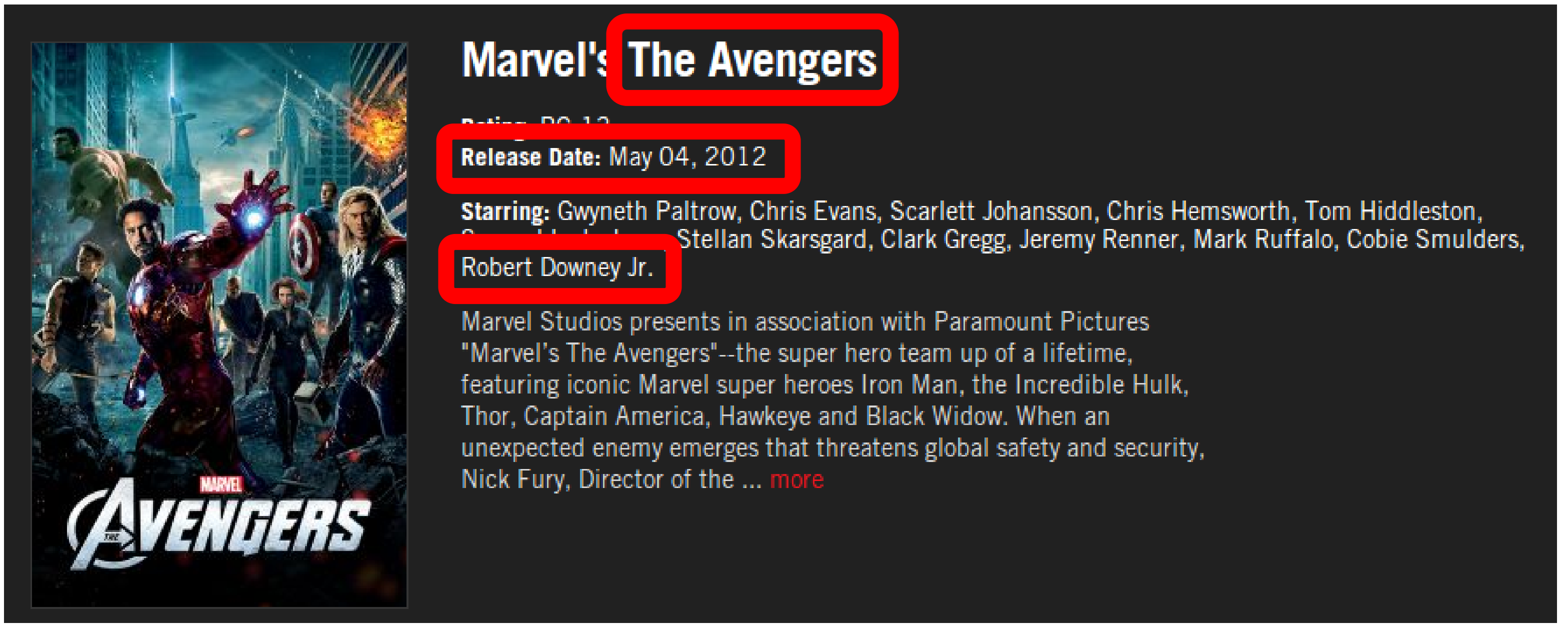
where

$$R = \bigcup_{e \in S} X_e$$

Relationship (as before) **Timestamps**

- Again, use co-occurrences for weights w_T

Co-occurrences on Web Scale



marvel.com

- *"Robert Downey Jr"* and *"May 4, 2012"* occurs 173 times on 71 different webpages
- US Release date of *The Avengers*
- Use MapReduce on 10B web pages (10k+ machines)

Complete Optimization Problem

- Generalized earlier **coverage** function to **linear combination** of **weighted coverage** functions

$$F(S) = \sum_{r \in R} w_R(r) + \sum_{e \in S} w_T(t_e)$$

where

$$R = \bigcup_{e \in S} X_e$$

- **Goal:** $\max_{|S| \leq k} F(S)$

- **Still NP-hard**
(because generalization of NP-hard problem)

Next

- How can we **actually optimize** this function?
- What **structure** is there that will help us do this efficiently?

- **Any questions so far?**

Approximate Solution

- **For this optimization problem, <u>Greedy</u> produces a solution S s.t.** $F(S) \geq (1-1/e)*OPT$ $(F(S) \geq 0.63*OPT)$
  [Nemhauser, Fisher, Wolsey '78]

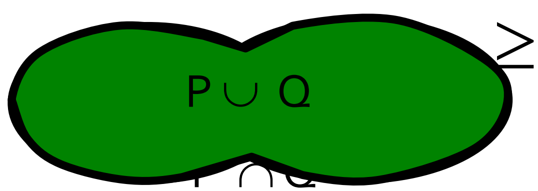
- **Claim holds for functions F*(·)* which are:**
  - ***Submodular, Monotone, Normal, Non-negative*** (discussed next)

Submodularity: Definition 1

Definition:

- Set function ***F(·)*** is called **submodular** if:
  For all ***P,Q*** $\subseteq$ ***U***:

$$F(P) + F(Q) \geq F(P \cup Q) + F(P \cap Q)$$



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Submodularity: Definition 2

- Checking the previous definition is not easy in practice
- Substitute $P = A \cup \{d\}$ and $Q = B$ where $\boxed{A \subseteq B \text{ and } d \notin B}$ in the definition above

From before: $F(P) + F(Q) \geq F(P \cup Q) + F(P \cap Q)$

$$F(A \cup \{d\}) + F(B) \geq F(A \cup \{d\} \cup B) + F((A \cup \{d\}) \cap B)$$

$$F(A \cup \{d\}) + F(B) \geq F(B \cup \{d\}) + F(A)$$

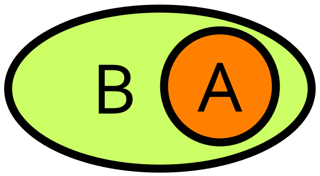
$$\boxed{F(A \cup \{d\}) - F(A) \geq F(B \cup \{d\}) - F(B)}$$

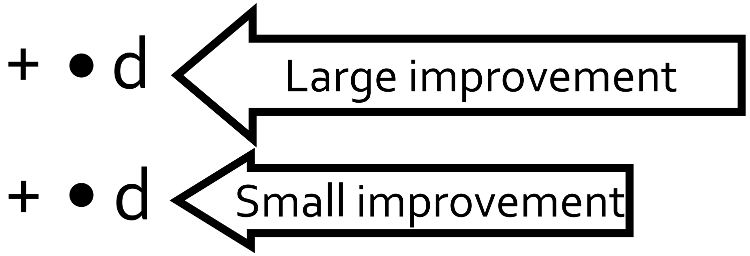
Common definition of Submodularity

Submodularity: Definition 2

- **Diminishing returns characterization**

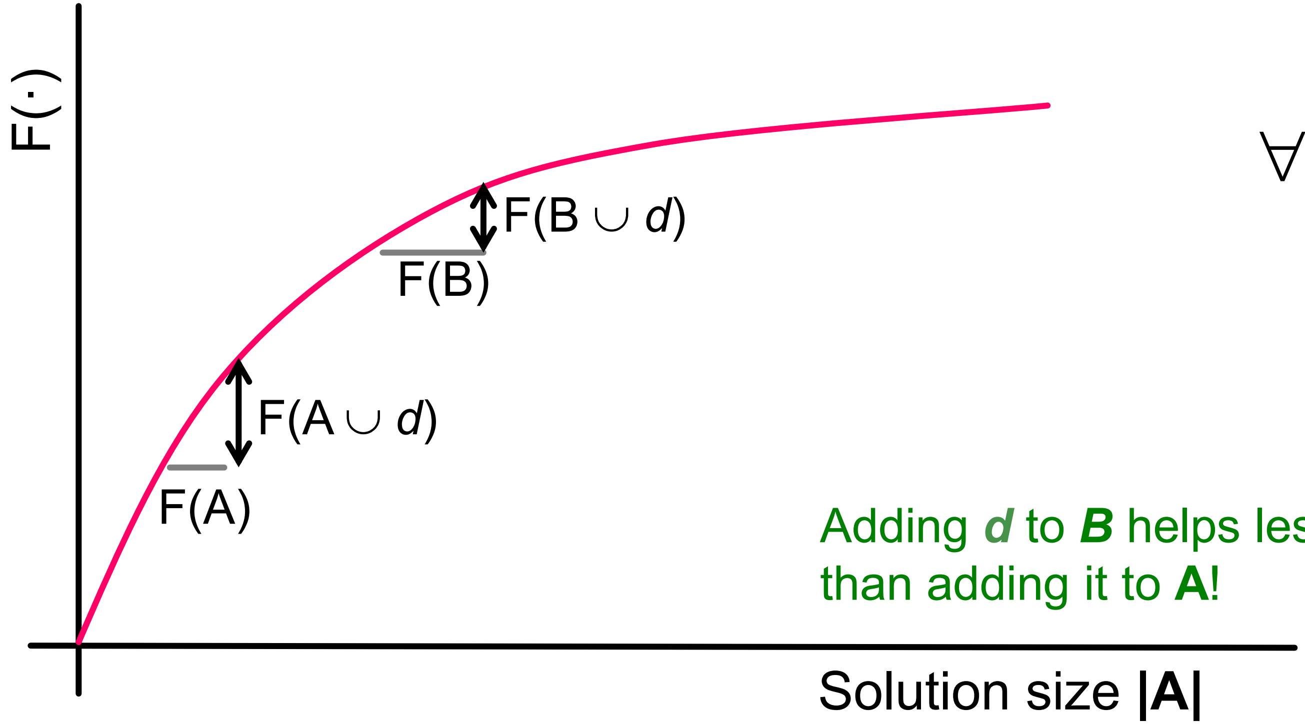
$$\underbrace{F(A \cup d) - F(A)}_{\text{Gain of adding } \boldsymbol{d} \text{ to a small set}} \geq \underbrace{F(B \cup d) - F(B)}_{\text{Gain of adding } \boldsymbol{d} \text{ to a large set}}$$





Submodularity: Diminishing Returns

$$\underbrace{F(A \cup d) - F(A)}_{\text{Gain of adding } \mathbf{d} \text{ to a small set}} \geq \underbrace{F(B \cup d) - F(B)}_{\text{Gain of adding } \mathbf{d} \text{ to a large set}}$$



$\forall A \subseteq B$

Adding ***d*** to ***B*** helps less than adding it to **A**!

Submodularity: An important property

Let **F_1 … F_M** be **submodular functions** and **λ_1 … $\lambda_M \geq 0$** and let **S** denote some solution set, then the non-negative linear combination **F(S)** (defined below) of these functions is also **submodular**.

$$F(S) = \sum_{i=1}^{M} \lambda_i F_i(S)$$

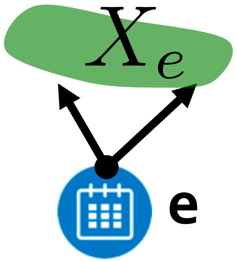
Submodularity: Approximation Guarantee

- **When maximizing a submodular function with cardinality constraints, Greedy produces a solution S for which *F(S)* $\geq$ *(1-1/e)*OPT***
  i.e., (*F(S)* $\geq$ *0.63*OPT*)
  [Nemhauser, Fisher, Wolsey '78]

- **Claim holds for functions F*(·)* which are:**
  - ***Monotone*:** if $A \subseteq B$ then $F(A) \leq F(B)$
  - ***Normal:*** ***F({})=0***
  - ***Non-negative:*** For any ***A***, ***F(A)*** $\geq$ ***0***
  - ***In addition to being submodular***

Back to our Timeline Problem

Simple Coverage Model



- Suppose we are given a set of events E
  - Each event **e** covers a set X_e of relationships **U**
- For a set of events $S \subseteq E$ we define:

$$F(S) = \left| \bigcup_{e \in S} X_e \right|$$

- Goal: We want to $\max_{|S| \leq k} F(S)$

Cardinality Constraint

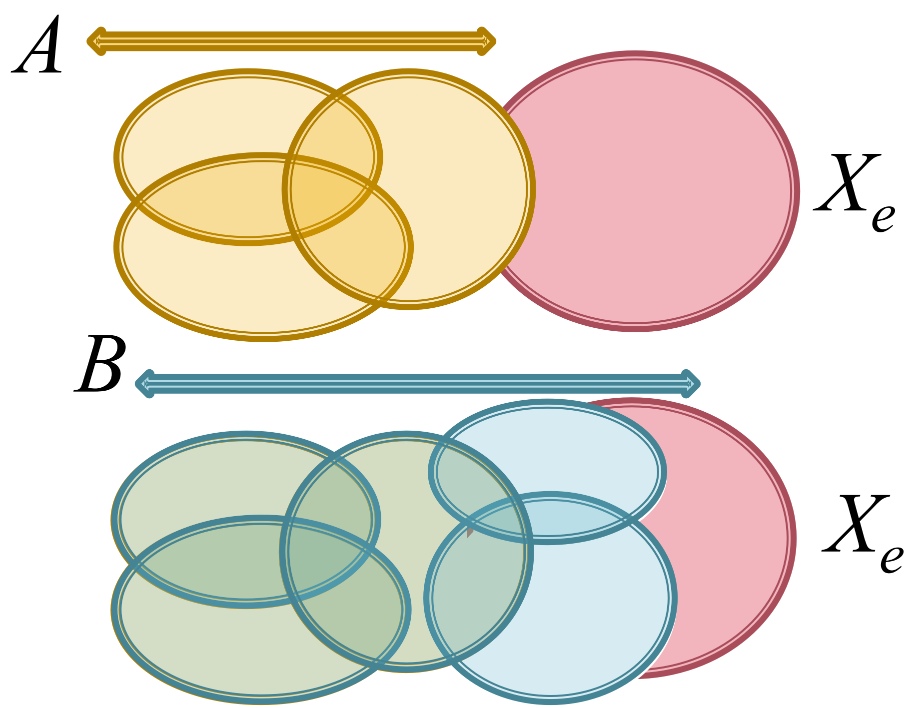
- Note: F(S) is a set function: $F(S) : 2^E \rightarrow \mathbb{N}$

Simple Coverage: Submodular?

- **Claim:** $F(S) = \left| \bigcup_{e \in S} X_e \right|$ **is submodular.**

Gain of adding X_e to a smaller set

Gain of adding X_e to a larger set



$$F(A \cup X_e) - F(A) \;\geq\; F(B \cup X_e) - F(B) \qquad \forall A \subseteq B$$

Simple Coverage: Other Properties

- **Claim:** $F(S) = \left| \bigcup_{e \in S} X_e \right|$ **is normal & monotone**
- **Normality:** When S is empty, $\bigcup_{e \in S} X_e$ is empty.
- **Monotonicity:** Adding a new event to S can never decrease the number of relationships covered by S.
- **What about non-negativity?**

Monotone: if ***A*** $\subseteq$ ***B*** then ***F*(A)** $\leq$ ***F*(B)**
Normal: F({})=0
Non-negative: For any ***A***, ***F(A)*** $\geq$ ***0***

Summary so far

	Simple Coverage	Weighted Coverage (Relationships)	Weighted Coverage (Timestamps)	Complete Optimization Problem
Submodularity	✓			
Monotonicity	✓			
Normality	✓			

Weighted Coverage (Relationships)

$$F(S) = \sum_{r \in R} w(r) \qquad w : R \to \mathbb{R}^+ \qquad \text{where } R = \bigcup_{e \in S} X_e$$

- **Claim: F(S) is submodular.**
  - Consider two sets A and B s.t. A $\subseteq$ B $\subseteq$ S and let us consider an event e $\notin$ B
  - Three possibilities when we add e to A or B:
    - **Case 1:** e does not cover any new relationships w.r.t both A and B

      F(A U {e}) – F(A) = 0 = F(B U {e}) – F(B)

Weighted Coverage (Relationships)

$$F(S) = \sum_{r \in R} w(r) \qquad w : R \rightarrow \mathbb{R}^+$$

- **Claim: F(S) is submodular.**
  - Three possibilities when we add e to A or B:
    - **Case 2:** e covers some new relationships w.r.t A but not w.r.t B

      F(A U {e}) – F(A) = v where $v \geq 0$

      F(B U {e}) – F(B) = 0

      Therefore, F(A U {e}) – F(A) ≥ F(B U {e}) – F(B)

Weighted Coverage (Relationships)

$$F(S) = \sum_{r \in R} w(r) \qquad w : R \to \mathbb{R}^+$$

- **Claim: F(S) is submodular.**
  - Three possibilities when we add e to A or B:
    - **Case 3:** e covers some new relationships w.r.t both A and B

      F(A U {e}) – F(A) = v where $v \geq 0$

      F(B U {e}) – F(B) = u where $u \geq 0$

      But, $v \geq u$ because e will always cover fewer new relationships w.r.t B than w.r.t A

Weighted Coverage (Relationships)

$$F(S) = \sum_{r \in R} w(r) \qquad w : R \to \mathbb{R}^+$$

where

$$R = \bigcup_{e \in S} X_e$$

- **Claim: F(S) is monotone and normal.**
- **Normality:** When S is empty, $R = \bigcup_{e \in S} X_e$ is empty.
- **Monotonicity:** Adding a new event to S can never decrease the number of relationships covered by S.

Summary so far

	Simple Coverage	Weighted Coverage (Relationships)	Weighted Coverage (Timestamps)	Complete Optimization Problem
Submodularity	✓	✓		
Monotonicity	✓	✓		
Normality	✓	✓		

Weighted Coverage (Timestamps)

$$F(S) = \sum_{e \in S} w_T(t_e)$$

- **Claim: F(S) is submodular, monotone and normal**
- Analogous arguments to that of weighted coverage (relationships) are applicable

Summary so far

	Simple Coverage	Weighted Coverage (Relationships)	Weighted Coverage (Timestamps)	Complete Optimization Problem
Submodularity	✓	✓	✓	
Monotonicity	✓	✓	✓	
Normality	✓	✓	✓	

Complete Optimization Problem

- Generalized earlier **coverage** function to non-negative **linear combination** of **weighted coverage** functions

$$F(S) = \boxed{F_1(S)} + \boxed{F_2(S)}$$

where

$$R = \bigcup_{e \in S} X_e$$

- **Goal:** $\max_{|S| \leq k} F(S)$

- **Claim: F(A) is submodular, monotone and normal**

Complete Optimization Problem

- **Submodularity:** F(S) is a non-negative linear combination of two submodular functions. Therefore, it is submodular too.
- **Normality:** $F_1(\{\}) = 0 = F_2(\{\})$
  $$F_1(\{\}) + F_2(\{\}) = 0$$
- **Monotonicity:** Let $A \subseteq B \subseteq S$,
  $$F_1(A) \leq F_1(B) \text{ and } F_2(A) \leq F_2(B)$$
  $$F_1(A) + F_2(A) \leq F_1(B) + F_2(B)$$

Summary so far

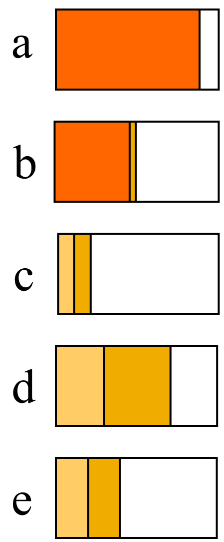
	Simple Coverage	Weighted Coverage (Relationships)	Weighted Coverage (Timestamps)	Complete Optimization Problem
Submodularity	✓	✓	✓	✓
Monotonicity	✓	✓	✓	✓
Normality	✓	✓	✓	✓

Lazy Optimization of Submodular Functions

Greedy Solution

Greedy

Marginal gain:
F(S∪x)-F(S)



Add element with highest marginal gain

- ***Greedy Algorithm is Slow!***
- At each iteration, we need to evaluate marginal gains of all the remaining elements
- Runtime O(|U| * K) for selecting K elements out of the set U

Speeding up Greedy

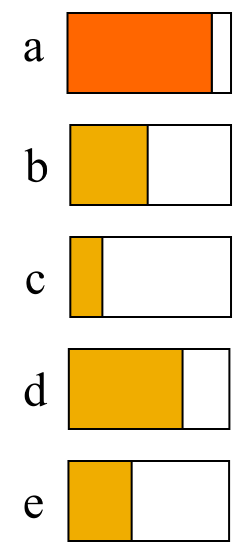
- ***In round i***:
  - So far we have $S_{i-1} = \{e_1 \dots e_{i-1}\}$
  - Now we pick an element $e \notin S_{i-1}$ which maximizes the marginal benefit $\Delta_i = F(S_{i-1} \cup \{e\}) - F(S_{i-1})$
- ***Key observation***:
  - ***Marginal gain of any element e can never increase!***
  - For every element e:
    $\Delta_i(e) \geq \Delta_j(e)$ for all iterations $i < j$

[Leskovec et al., KDD ’07]

Lazy Greedy

- **Idea:**
  - Use Δ_i as upper-bound on Δ_j ($j > i$)
- **Lazy Greedy:**
  - Keep an ordered list of marginal benefits Δ_i from previous iteration
  - Re-evaluate Δ_i only for top node
  - Re-sort and prune

Upper bound on Marginal gain Δ_1



A_1={a}

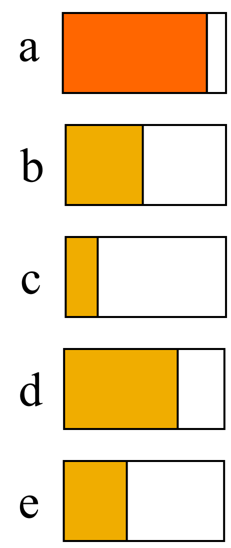
$$F(A \cup \{d\}) - F(A) \;\geq\; F(B \cup \{d\}) - F(B) \qquad A \subseteq B$$

[Leskovec et al., KDD '07]

Lazy Greedy

- **Idea:**
  - Use Δ_i as upper-bound on Δ_j ($j > i$)
- **Lazy Greedy:**
  - Keep an ordered list of marginal benefits Δ_i from previous iteration
  - Re-evaluate Δ_i only for top node
  - Re-sort and prune

Upper bound on Marginal gain Δ_2



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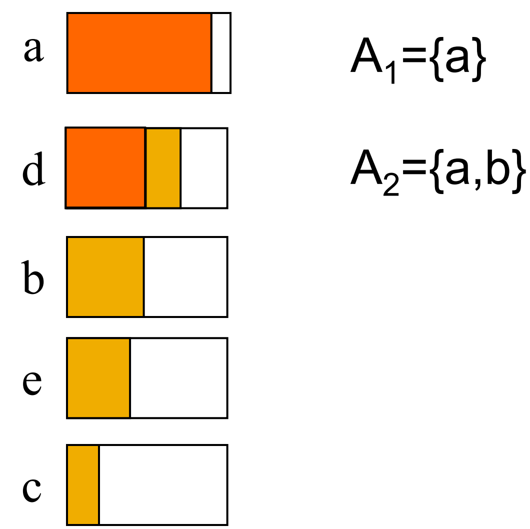

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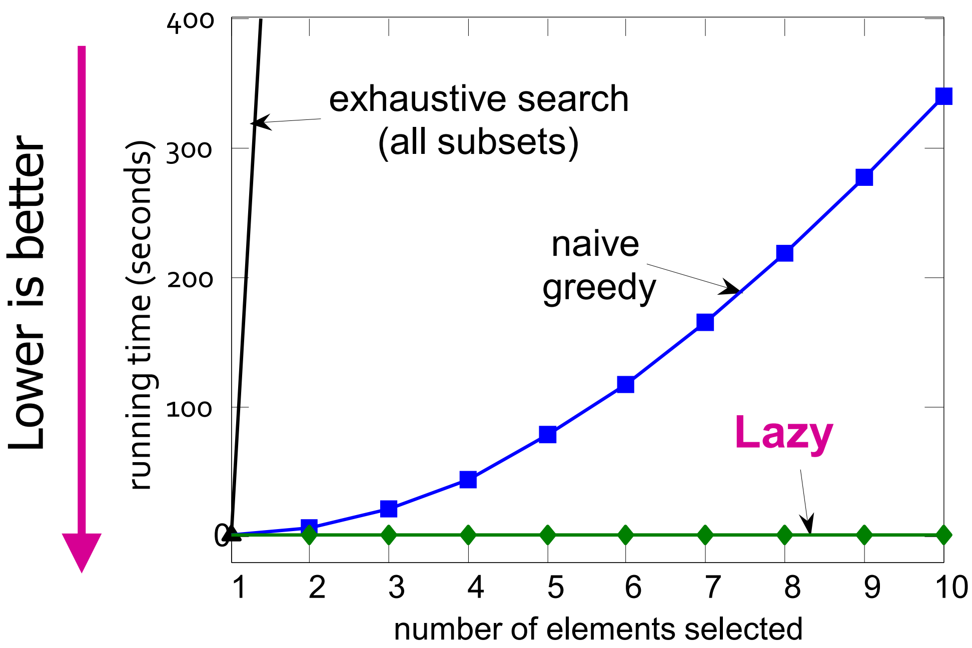
Speed Up of Lazy Greedy Algorithm

- Lazy greedy offers significant speed-up over traditional greedy implementations in practice.



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