Remember: No lecture next Tuesday – extra TA office hours for projects instead

Mining Data Streams (Part 2)

CS547 Machine Learning for Big Data
Tim Althoff
PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Today's Lecture

- **More algorithms for streams:**
  - **(1) Filtering a data stream: Bloom filters**
    - Select elements with property **x** from stream
  - **(2) Counting distinct elements: Flajolet-Martin**
    - Number of distinct elements in the last ***k*** elements of the stream
  - **(3) Estimating moments: AMS method**
    - Estimate std. dev. of last ***k*** elements

(1) Filtering Data Streams

Filtering Data Streams

- **Each element of data stream is a tuple**
- Given a list of keys **S**
- **Determine which tuples of stream are in *S***

- **Obvious solution: Hash table**
  - But suppose we **do not have enough memory** to store all of ***S*** in a hash table
    - E.g., we might be processing millions of filters on the same stream

Applications

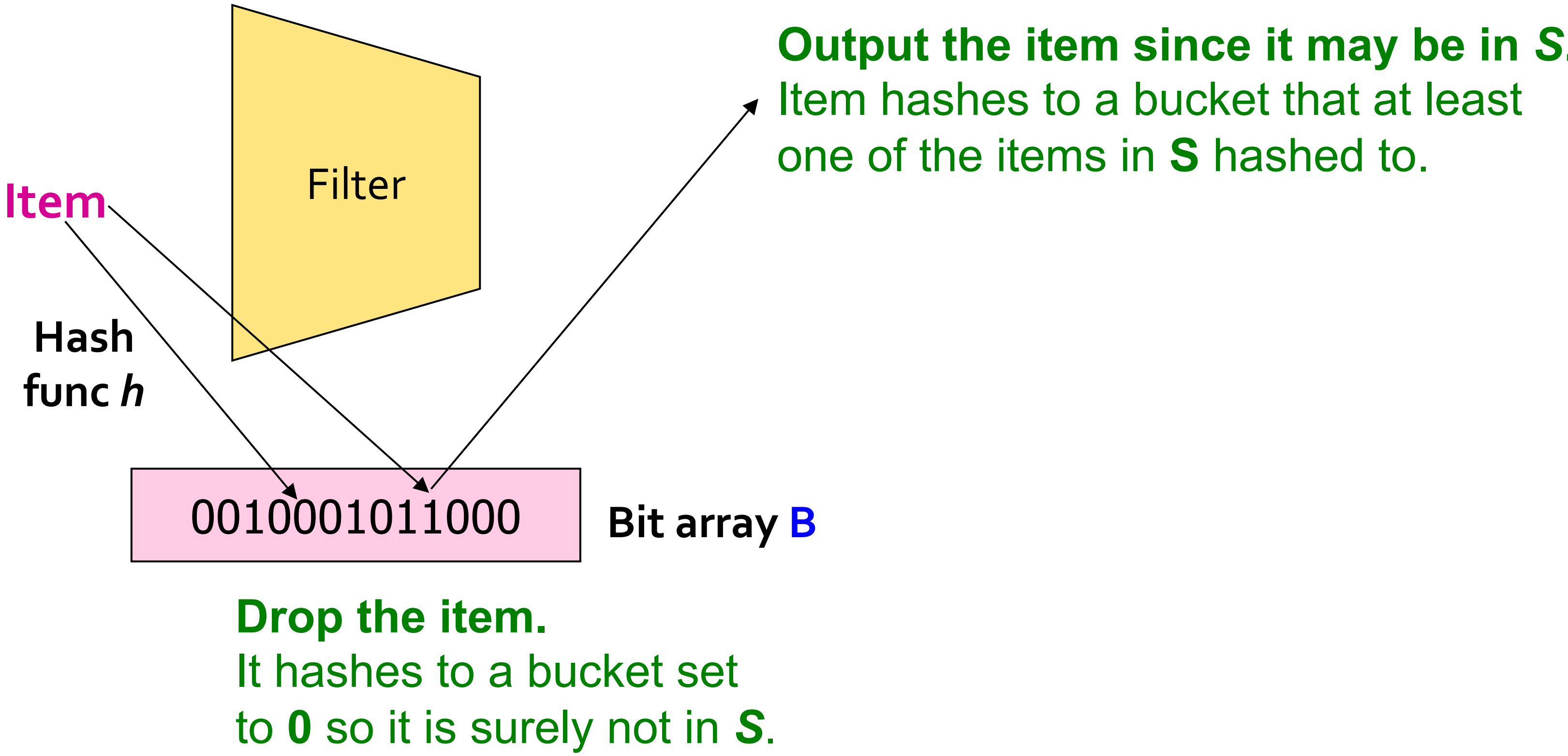
- **Example: Email spam filtering**
  - We know 1 billion “good” email addresses
    - Or, each user has a list of trusted addresses
  - If an email comes from one of these, it is **NOT** spam
- **Publish-subscribe systems**
  - You are collecting lots of messages (news articles)
  - People express interest in certain sets of keywords
  - Determine whether each message matches user’s interest
- **Content filtering:**
  - You want to make sure the user does not see the same ad multiple times
- **Web cache filtering:**
  - Has this piece of content been requested before? Then cache it now.

First Cut Solution (1)

Given a set of keys *S* that we want to filter

- Create a **bit array *B*** of ***n*** bits, initially all ***0*s**
- Choose a **hash function *h*** with range **[*0,n*)**
- Hash each member of $s \in S$ to one of *n* buckets, and set that bit to **1**, i.e., ***B[h(s)]=1***
- Hash each element ***a*** of the stream and output only those that hash to bit that was set to **1**
  - **Output *a* if B[h(a)] == 1**

First Cut Solution (2)



- **Creates false positives but no false negatives**
  - If the item is in ***S*** we surely output it, if not we may still output it

First Cut Solution (3)

- **|S| = 1 billion email addresses**
  |B|= 1GB = 8 billion bits
- If the email address is in ***S***, then it surely hashes to a bucket that has the bit set to **1**, so it always gets through (***no false negatives***)
- Approximately **1/8** of the bits are set to **1**, so about **1/8th** of the addresses not in ***S*** get through to the output (***false positives***)
  - Actually, less than **1/8th**, because more than one address might hash to the same bit

Analysis: Throwing Darts (1)

- **More accurate analysis for the number of false positives**

- **Consider:** If we throw ***m*** darts into ***n*** equally likely targets, **what is the probability that a target gets at least one dart?**

- **In our case:**
  - **Targets** = bits/buckets
  - **Darts** = hash values of items

Analysis: Throwing Darts (2)

- We have ***m*** darts, ***n*** targets
- **What is the probability that a target gets at least one dart?**

Equals **1/e** as **n** $\rightarrow\infty$

Equivalent

$$1 - \left(1 - 1/n\right)^{n\left(m/n\right)}$$

$$1 - e^{-m/n}$$

Probability some target **X** not hit by a dart

Probability at least one dart hits target **X**

Approximation is especially accurate when n is large

Analysis: Throwing Darts (3)

- **Fraction of 1s in the array B =**
  = probability of false positive = $1 - e^{-m/n}$

- **Example:** $\mathbf{10^9}$ darts, $\mathbf{8 \cdot 10^9}$ targets
  - Fraction of **1s** in **B = $1 - e^{-1/8}$ = 0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

Bloom Filter

- Consider: **|S| = m, |B| = n**
- Use k independent hash functions $h_1, \ldots, h_k$
- **Initialization:**
  - Set **B** to all **0s**
  - Hash each element $s \in S$ using each hash function h_i, set **B[$h_i(s)$] = 1** (for each $i = 1,.., k$)

(**note:** we have a single array B!)

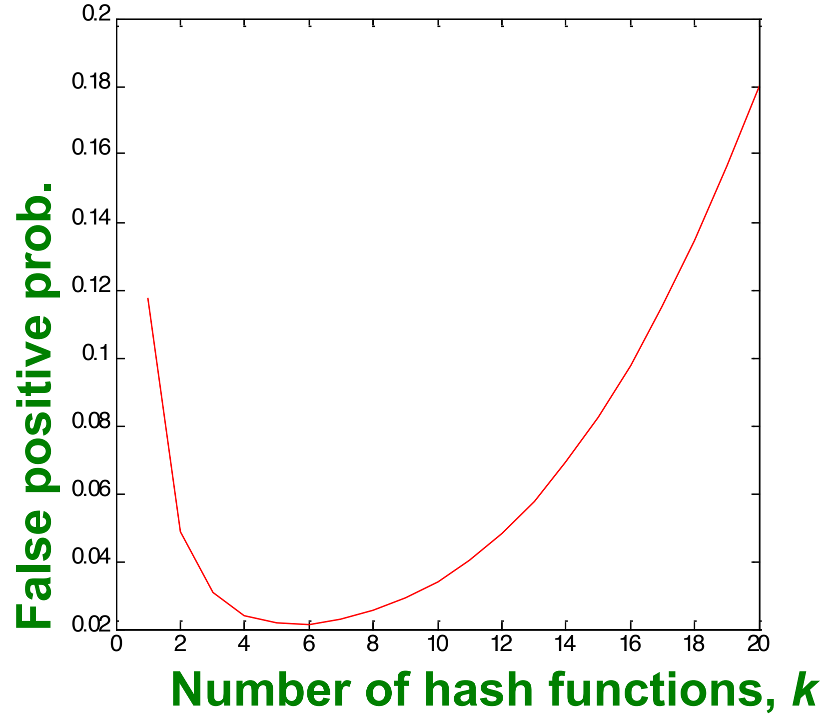
- **Run-time:**
  - When a stream element with key x arrives
    - If **B[$h_i(x)$] = 1** for all i = 1,..., k then declare that x is in S
      - That is, x hashes to a bucket set to **1** for every hash function $h_i(x)$
    - Otherwise discard the element x

Bloom Filter – Analysis

- **What fraction of the bit vector B are 1s?**
  - Throwing ***k·m*** darts at ***n*** targets
  - So fraction of **1**s is $(1 - e^{-km/n})$
- But we have ***k*** independent hash functions and we only let the element ***x*** through **if all *k*** hash element ***x*** to a bucket of value **1**
- So, **false positive probability** = $(1 - e^{-km/n})^k$

Bloom Filter – Analysis (2)

- ***m* = 1 billion, *n* = 8 billion**
  - **k = 1**: $(1 - e^{-1/8})$ = **0.1175**
  - **k = 2**: $(1 - e^{-1/4})^2$ = **0.0493**



- **What happens as we keep increasing *k*?**

- Optimal value of ***k***: ***n/m* ln(2)**
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-3/4})^6$ = **0.0216**

Optimal *k*: *k* which gives the lowest false positive probability

Bloom Filter: Wrap-up

- Bloom filters allow for filtering / set membership
- **Bloom filters guarantee no false negatives, and use limited memory**
  - Great for pre-processing before more expensive checks
- **Suitable for hardware implementation**
  - Hash function computations can be parallelized

- Is it better to have **1** big **B** or ***k*** small **B**s?
  - **It is the same:** $(1 - e^{-km/n})^k$ vs. $(1 - e^{-m/(n/k)})^k$
  - But keeping **1 big B** is simpler

(2) Counting Distinct Elements

Counting Distinct Elements

- **Problem:**
  - Data stream consists of a universe of elements chosen from a set of size ***N***
  - Maintain a count of the number of distinct elements seen so far
- **Obvious approach:**
  Maintain the set of elements seen so far
  - That is, keep a hash table of all the distinct elements seen so far

Applications

- **How many different words are found among the Web pages being crawled at a site?**
  - Unusually low or high numbers could indicate artificial pages (spam?)
- **How many different Web pages does each customer request in a week?**
- **How many distinct products have we sold in the last week?**

Using Small Storage

- **Real problem:** **What if we do not have space to maintain the set of elements seen so far?**
- **Estimate the count in an unbiased way**
- **Accept that the count may have a little error, but limit the probability that the error is large**

Flajolet-Martin Approach

- Pick a hash function ***h*** that maps each of the ***N*** elements to at least $\log_2 N$ bits
- For each stream element ***a***, let ***r*(*a*)** be the number of trailing **0s** in ***h*(*a*)**
  - **r(a)** = position of first 1 counting from the right
    - E.g., say ***h(a) = 12***, then ***12*** is ***1100*** in binary, so ***r(a) = 2***
- Record ***R* = the maximum *r*(*a*) seen**
  - $\mathbf{R = max_a\ r(a)}$, over all the items ***a*** seen so far
- **Estimated number of distinct elements = 2^R**

Why It Works: Intuition

- **Rough intuition why Flajolet-Martin works:**
  - ***h(a)*** hashes ***a*** with **equal prob.** to any of ***N*** values
  - Then ***h(a)*** is a sequence of **$\log_2 N$** bits, where 2^{-r} fraction of all ***a***s have a tail of ***r*** zeros
    - About 50% of ***a***s hash to *****0**
    - About 25% of ***a***s hash to ****00**
    - So, if we saw the longest tail of ***r=2*** (i.e., item hash ending ***100**) then we have probably seen **about *4*** distinct items so far
  - **So, it takes to hash about 2^r items before we see one with zero-suffix of length *r***

Why It Works: More formally

- **Now we show why Flajolet-Martin works**
- **Formally, we will show that probability of finding a tail of *r* zeros:**
  - **Goes to 1 if $m \gg 2^r$**
  - **Goes to 0 if $m \ll 2^r$**

  where m is the number of distinct elements seen so far in the stream
- **Thus, 2^R will almost always be around *m*!**

Why It Works: More formally

- **What is the probability that a given *h*(*a*) ends in at least *r* zeros? It is 2^{-r}**
  - **h(a)** hashes elements uniformly at random
  - Probability that a random number ends in at least ***r*** zeros is $\mathbf{2^{-r}}$
- Then, the probability of **NOT** seeing a tail of length ***r*** among ***m*** distinct elements:

$$\boxed{\left(\boxed{1-2^{-r}}\right)^{m}}$$

Prob. all m elements end in fewer than ***r*** zeros.

Prob. that given **h(a)** ends in fewer than ***r*** zeros

Why It Works: More formally

- **Note:** $(1-2^{-r})^m = (1-2^{-r})^{2^r (m2^{-r})} \approx e^{-m2^{-r}}$
- **Prob. of NOT finding a tail of length *r* is:**
  - If ***m << 2^r^***, then prob. tends to **1**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 1$ as **$m/2^r \to 0$**
    - So, the probability of finding a tail of length ***r*** tends to **0**
  - If ***m >> 2^r^***, then prob. tends to **0**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 0$ as **$m/2^r \to \infty$**
    - So, the probability of finding a tail of length ***r*** tends to **1**
- **Thus, 2^R will almost always be around *m*!**

Why It Doesn't Work

- **$E[2^R]$ is actually infinite**
  - Observing R has some probability
  - Probability halves when $\boldsymbol{R} \rightarrow \boldsymbol{R+1}$, but value doubles
  - Each possible large R contributes to exp. value
- **Workaround involves using many hash functions h_i and getting many samples of R_i**
- **How are samples R_i combined?**
  - **Average?** What if one very large value 2^{R_i}?
  - **Median?** All estimates are a power of **2**
  - **Solution:**
    - Partition your samples into small groups
    - Take the median of groups
    - Then take the average of the medians

(3) Computing Moments

Generalization: Moments

- **Suppose a stream has elements chosen from a set *A* of *N* values**

- **Let m_i be the *number of times* value *i* occurs in the stream**

- **The k^{th} (frequency) *moment* is**

$$\sum_{i \in A} (m_i)^k$$

This is the same way as moments are defined in statistics. But there one typically "centers" the moment by subtracting the mean.

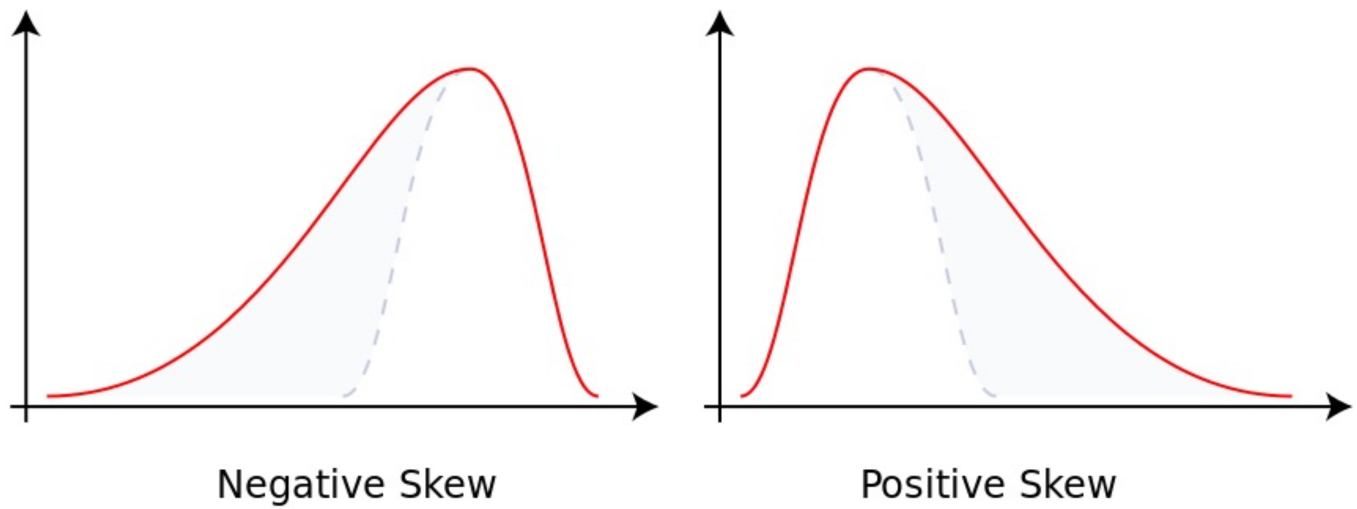
Special Cases

$$\sum_{i \in A} (m_i)^k$$

- **0th moment =** number of distinct elements
  - The problem just considered
- **1st moment =** count of the numbers of elements = length of the stream
  - Easy to compute, so not particularly useful
- **2nd moment =** ***surprise number S*** **=** a measure of how uneven the distribution is
  - Very useful

Moments

- **Third Moment is Skew:**



- **Fourth moment: Kurtosis**
  - peakedness (width of peak), tail weight, and lack of shoulders (distribution primarily peak and tails, not in between).

Example: Surprise Number

- **Measure of how uneven the distribution is**

- **Stream of length 100**
- **11 distinct values**

- Item **counts m_i: 10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9**
  Surprise *S* = 910

- Item **counts m_i : 90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1**
  Surprise *S* = 8,110

AMS Method

- **AMS method works for all moments**
- **Gives an unbiased estimate**
- **We will just concentrate on the 2nd moment**
  - Will generalize later
- **We pick and keep track of many variables *X*:**
  - For each variable ***X*** we store ***X.el*** and ***X.val***
    - ***X.el*** corresponds to the item ***i***
    - ***X.val*** corresponds to the **count** m_i of item ***i***
  - Note this requires a count in main memory, so number of ***X***s is limited
- **Our goal is to compute** $S = \sum_i m_i^2$

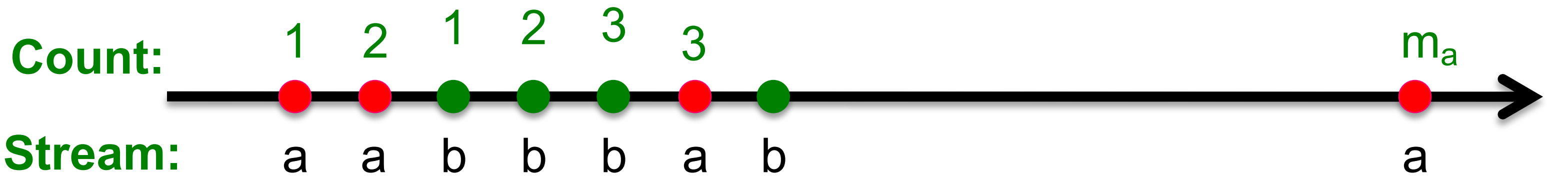
One Random Variable (X)

- **How to set *X.val* and *X.el*?**
  - Assume stream has length ***n*** (we relax this later)
  - Pick some random time ***t*** (***t<n***) to start, so that any time is equally likely
  - Let at time ***t*** the stream have item ***i***. ***We set X.el = i***
  - Then we maintain count ***c*** (***X.val = c***) of the number of ***is*** in the stream starting from the chosen time ***t***
- **Then the estimate of the 2nd moment ($\sum_i m_i^2$) is:**

$$S = f(X) = n\,(2 \cdot c - 1)$$

  - Note, we will keep track of multiple **X**s, ($\mathbf{X_1}, \mathbf{X_2}, \ldots \mathbf{X_k}$) and our final estimate will be $S = 1/k \sum_j^k f(X_j)$

Expectation Analysis



- **2nd moment is $S = \sum_i m_i^2$**
- c_t … number of times item at time t appears from time **t** onwards ($c_1 = m_a$, $c_2 = m_a - 1$, $c_3 = m_b$)
- $E[f(X)] = \frac{1}{n} \sum_{t=1}^{n} n(2c_t - 1)$

$$= \frac{1}{n} \sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$$

m_i … total count of item i in the stream (we are assuming stream has length **n**)

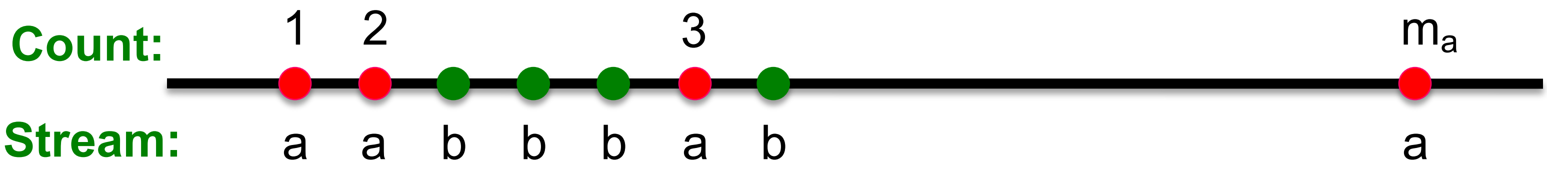
Group times by the value seen

Time t when the last i is seen (c_t=1)

Time t when the penultimate i is seen (c_t=2)

Time t when the first i is seen (c_t=m_i)

Expectation Analysis



- $E[f(X)] = \frac{1}{n}\sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$
  - Little side calculation: $(1 + 3 + 5 + \cdots + 2m_i - 1) = \sum_{i=1}^{m_i}(2i-1) = 2\frac{m_i(m_i+1)}{2} - m_i = (m_i\ \ )^2$
- **Then $\boldsymbol{E[f(X)] = \frac{1}{n}\sum_i n\,(m_i)^2}$**

- **So, $\mathbf{E[f(X)]} = \boldsymbol{\sum_i (m_i)^2 = S}$**
- **We have the second moment (in expectation)!**

Higher-Order Moments

- **For estimating k^{th} moment we essentially use the same algorithm but change the estimate f(X):**
  - For **k=2** we used ***n* (2·c – 1)**
  - For **k=3** we use: ***n* (3·c² – 3c + 1)** (where **c=X.val**)
- **Why?**
  - **For k=2:** Remember we had $(1 + 3 + 5 + \cdots + 2m_i - 1)$ and we showed terms ***2c-1*** (for **c=1,…,m**) sum to ***m²***
    - $\sum_{c=1}^{m}(2c-1) = \sum_{c=1}^{m} c^2 - \sum_{c=1}^{m}(c-1)^2 = m^2$
    - **So:** $\boldsymbol{2c - 1 = c^2 - (c-1)^2}$
  - **For k=3:** $\mathbf{c^3 - (c-1)^3 = 3c^2 - 3c + 1}$
- **Generally:** Estimate f(X) $= n\,(c^k - (c-1)^k)$

Combining Samples

- **In practice:**
  - Compute $f(X) = n(2c - 1)$ for as many variables ***X*** as you can fit in memory
  - Average them in groups
  - Take median of averages
- **Problem: Streams never end**
  - We assumed there was a number ***n***, the number of positions in the stream
  - But real streams go on forever, so ***n*** is a variable – the number of inputs seen so far

Streams Never End: Fixups

- **(1)** The variables ***X*** have ***n*** as a factor – keep ***n*** separately; just hold the count in ***X***
- **(2)** Suppose we can only store ***k*** counts. We must throw some ***X***s out as time goes on:
  - **Objective:** Each starting time ***t*** is selected with probability ***k/n***
  - **Solution: (fixed-size / reservoir sampling!)**
    - Choose the first ***k*** times for ***k*** variables
    - When the n^{th} element arrives (***n > k***), choose it with probability ***k/n***
    - If you choose it, throw one of the previously stored variables **X** out, with equal probability

Problems on Data Streams

- **Filtering a data stream**
  - Select elements with property ***x*** from the stream
- **Counting distinct elements**
  - Number of distinct elements in the last ***k*** elements of the stream
- **Estimating moments**
  - Estimate avg./std. dev. of elements in stream

- **Remember: No lecture next Tuesday – Project Group meetings instead**