Case Study 1: Estimating Click Probabilities

Intro
Logistic Regression
Gradient Descent + SGD

Machine Learning for Big Data
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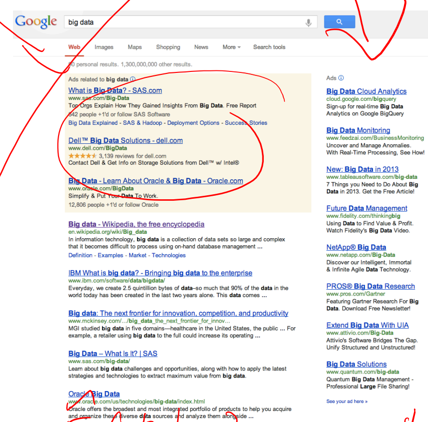
Announcements:

- Lecture 2 cancelled
  - TAs will hold a python recitation
- HW1 posted today.
- (starting NEXT week) TA office hours
- Readings: please do them.
- Project Proposals: please start thinking about it!

- Today:
  - Review: click prediction and logistic regression

Ad Placement Strategies

banner



- Companies bid on ad prices

  $c_1 = \$10$ $\quad c_3 = \$100$

  $c_2 = \$20$

- Which ad wins? (many simplifications here)
  - Naively:

    $c_3 = \$100$

  - But:

    paid on clicks

  - Instead:

    $\hat{Pr}(\text{click} \mid c_3) = 0.01$

    $\hat{Pr}(\text{click} \mid c_2) = 0.1$

$E[\$ \mid c_1] = 0.01 \times 100 = \$1$

$\hat{E}[\$ \mid c_2] = 0.1 \times 20 = \$2$

Key Task: Estimating Click Probabilities

given some keywords + context.

- What is the probability that user *i* will click on ad *j*

- Not important just for ads:
  - Optimize search results
  - Suggest news articles
  - Recommend products

- Methods much more general, useful for:
  - Classification
  - Regression
  - Density estimation

Learning Problem for Click Prediction

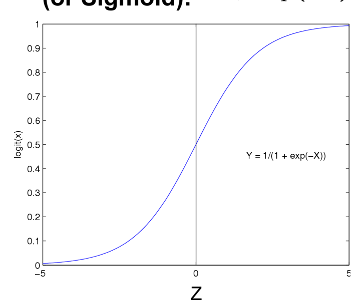
- Prediction task: $Y \in \{0,1\}$ $Pr(Y=1|X)$
  Y is a click.
- Features:
  $X =$ (features of ad, features of person, keyword, person index, other context)
- Data:
  $\{(x^i, y^i)\}$
  - Batch:
    fixed dataset $(x^1, y^1), \ldots, (x^n, y^n)$
  - Online:
    data as a stream
    when user arrives at time t
    show ad, observe Y.
- Many approaches (e.g., logistic regression, SVMs, naïve Bayes, decision trees, boosting,…)
  - Focus on logistic regression; captures main concepts, ideas generalize to other approaches

Logistic Regression

- Learn P(Y|**X**) directly
  - Assume a particular functional form
  - Sigmoid applied to a linear function of the data:

$$P(Y=0|X,W) = \frac{1}{1+exp(\underbrace{w_0+\sum_i w_i X_i}_{z})}$$

Logistic function (or Sigmoid): $\frac{1}{1+exp(-z)}$



Features can be discrete or continuous!

Very convenient!

$\Sigma w_i X_i > 0$

$w_0 +$

$$P(Y=0\,|X=<X_1,...X_n>) = \frac{1}{1+exp(w_0+\sum_i w_i X_i)}$$

1 0

linear classification rule!

implies

$$\ln \frac{P(Y=1\,|X)}{P(Y=0\,|X)} = w_0 + \sum_i w_i X_i$$

log odds

Digression: Logistic regression more generally

- Logistic regression in more general case, where Y in $\{y_1,\ldots,y_R\}$

softmax

for $k<R$

$$P(Y=y_k|X) = \frac{\exp(w_{k0}+\sum_{i=1}^n w_{ki}X_i)}{1+\sum_{j=1}^{R-1}\exp(w_{j0}+\sum_{i=1}^n w_{ji}X_i)}$$

for $k=R$ (normalization, so no weights for this class)

$$P(Y=y_R|X) = \frac{1}{1+\sum_{j=1}^{R-1}\exp(w_{j0}+\sum_{i=1}^n w_{ji}X_i)}$$

Features can be discrete or continuous!

Loss function: Conditional Likelihood

- Have a bunch of iid data of the form:

$(x^i, y^i)_{i=1:N} \triangleq D = (D_x, D_y)$

- Discriminative (logistic regression) loss function:

Conditional Data Likelihood

iid

$\arg\max_w \Pr(D_y \mid D_x, w) = \arg\max_w \prod_{j=1}^{N} \Pr(y^j \mid x^j, w)$

$= \arg\max_w \log \prod_{j=1}^{N} \Pr(y^j \mid x^j, w)$

//

$$\ln P(\mathcal{D}_Y \mid \mathcal{D}_{\mathbf{X}}, \mathbf{w}) = \sum_{j=1}^{N} \ln P(y^j \mid \mathbf{x}^j, \mathbf{w})$$

Expressing Conditional Log Likelihood

$$P(Y=0|\mathbf{X},\mathbf{w}) = \frac{1}{1+exp(w_0+\sum_i w_i X_i)}$$

$$P(Y=1|\mathbf{X},\mathbf{w}) = \frac{exp(w_0+\sum_i w_i X_i)}{1+exp(w_0+\sum_i w_i X_i)}$$

$$l(\mathbf{w}) \equiv \sum_j \ln P(y^j|\mathbf{x}^j,\mathbf{w})$$

$= \sum_j \{ \log \Pr(y=1|x^j, w)$ if $y^j = 1$

$\log \Pr(y=0|x^j, w) \}$ if $y^j = 0$

$$\ell(\mathbf{w}) = \sum_j y^j \ln P(Y=1|\mathbf{x}^j,\mathbf{w}) + (1-y^j)\ln P(Y=0|\mathbf{x}^j,\mathbf{w})$$

$$= \sum_j y^j (w_0 + \sum_{i=1}^{d} w_i x_i^j) - \ln\left(1 + \exp(w_0 + \sum_{i=1}^{d} w_i x_i^j)\right)$$

[illegible]

Maximizing Conditional Log Likelihood

concave not

$$l(\mathbf{w}) \equiv \ln \prod_j P(y^j|\mathbf{x}^j, \mathbf{w})$$
$$= \sum_j y^j(w_0 + \sum_{i=1}^d w_i x_i^j) - \ln\left(1 + \exp(w_0 + \sum_{i=1}^d w_i x_i^j)\right)$$

iterative methods

Good news: $l(\mathbf{w})$ is concave function of $\mathbf{w}$, no local optima problems

Bad news: no closed-form solution to maximize $l(\mathbf{w})$

Good news: concave functions easy to optimize

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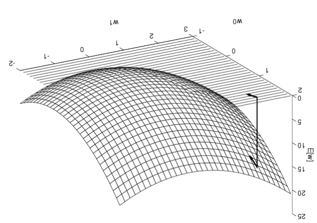
Optimizing concave function – Gradient ascent

- Conditional likelihood for logistic regression is *concave*
- Find optimum with *gradient ascent*



Gradient: $\nabla_{\mathbf{w}} l(\mathbf{w}) = [\frac{\partial l(\mathbf{w})}{\partial w_0}, \ldots, \frac{\partial l(\mathbf{w})}{\partial w_n}]'$

for small η

l will increase

Step size, $\eta>0$

Update rule: $\Delta\mathbf{w} = \eta \nabla_{\mathbf{w}} l(\mathbf{w})$

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \frac{\partial l(\mathbf{w})}{\partial w_i}$$

- Gradient ascent is simplest of optimization approaches
  - e.g., Conjugate gradient ascent much better (see reading)

L-BFGS

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Gradient Ascent for LR

Gradient ascent algorithm: iterate until change < ε

$$w_0^{(t+1)} \leftarrow w_0^{(t)} + \eta \sum_j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})]$$

For $i = 1,\ldots,d$,

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})]$$

$$\vec{w} \leftarrow \vec{w} + \eta \sum_j \vec{x}^j (y^j - \hat{p}^j)$$

repeat

Regularized Conditional Log Likelihood

- If data are linearly separable, weights go to infinity
- Leads to overfitting → Penalize large weights

$$\sum_{j=1}^{d} w_j^2$$

- Add regularization penalty, e.g., L_2:

$$\ell(\mathbf{w}) = \ln \prod_j P(y^j | \mathbf{x}^j, \mathbf{w})) - \frac{\lambda}{2}||\mathbf{w}||_2^2$$

- Practical note about w_0: don't regularize offset

Standard v. Regularized Updates

- Maximum conditional likelihood estimate

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[\prod_{j=1}^{N} P(y^j \mid \mathbf{x}^j, \mathbf{w}) \right]$$

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})]}$$

- Regularized maximum conditional likelihood estimate

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[\prod_j P(y^j | \mathbf{x}^j, \mathbf{w})) \right] - \frac{\lambda}{2} \sum_{i>0} w_i^2$$

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})] \right\}}$$

Stopping criterion

$\gamma \geq \lambda$

$$\ell(\mathbf{w}) = \ln \prod_j P(y^j | \mathbf{x}^j, \mathbf{w})) - \frac{\lambda}{2} ||\mathbf{w}||_2^2$$

- Regularized logistic regression is strongly concave
  - Negative second derivative bounded away from zero:

f(x) strongly concave (and f'' exists) $\longleftrightarrow$ $-f''(x) \geq \gamma$ for $\gamma > 0$

- Strong concavity (convexity) is super helpful!!
- For example, for strongly concave *l*(**w**):

$$\ell(\mathbf{w}^*) - \ell(\mathbf{w}) \leq \frac{1}{2\lambda} ||\nabla \ell(\mathbf{w})||_2^2$$

Convergence rates for gradient descent/ascent

- Number of iterations to get to accuracy

$$\ell(\mathbf{w}^*) - \ell(\mathbf{w}) \leq \epsilon$$

- If func l(w) Lipschitz: $O(1/\epsilon^2)$
- If gradient of func Lipschitz: $O(1/\epsilon)$
- If func is strongly convex: $O(\ln(1/\epsilon))$

Challenge 1: Complexity of computing gradients

- What's the cost of a gradient update step for LR???

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})] \right\}}$$

Challenge 2: Data is streaming

- Assumption thus far: **Batch data**
- But, click prediction is a streaming data task:
  - User enters query, and ad must be selected:
    - Observe $\mathbf{x}^j$, and must predict y^j
  - User either clicks or doesn't click on ad:
    - Label y^j is revealed afterwards
      - Google gets a reward if user clicks on ad
  - Weights must be updated for next time:

Learning Problems as Expectations

- Minimizing loss in training data:
  - Given dataset:
    - Sampled iid from some distribution p(**x**) on features:
  - Loss function, e.g., hinge loss, logistic loss,…
  - We often minimize loss in training data:

$$\ell_{\mathcal{D}}(\mathbf{w}) = \frac{1}{N}\sum_{j=1}^{N}\ell(\mathbf{w},\mathbf{x}^j)$$

- However, we should really minimize expected loss on all data:

$$\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = \int p(\mathbf{x})\ell(\mathbf{w},\mathbf{x})d\mathbf{x}$$

- So, we are approximating the integral by the average on the training data

Gradient Ascent in Terms of Expectations

- "True" objective function:

$$\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = \int p(\mathbf{x})\ell(\mathbf{w},\mathbf{x})d\mathbf{x}$$

- Taking the gradient:
- "True" gradient ascent rule:
- How do we estimate expected gradient?

SGD: Stochastic Gradient Ascent (or Descent)

- "True" gradient: $\nabla\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\nabla\ell(\mathbf{w},\mathbf{x})\right]$
- Sample based approximation:
- What if we estimate gradient with just one sample???
  - Unbiased estimate of gradient
  - Very noisy!
  - Called stochastic gradient ascent (or descent)
    - Among many other names
  - VERY useful in practice!!!

Stochastic Gradient Ascent: General Case

- Given a stochastic function of parameters:
  - Want to find maximum
- Start from $\mathbf{w}^{(0)}$
- Repeat until convergence:
  - Get a sample data point $\mathbf{x}^t$
  - Update parameters:
- Works in the online learning setting!
- Complexity of each gradient step is constant in number of examples!
- In general, step size changes with iterations

Stochastic Gradient Ascent for Logistic Regression

- Logistic loss as a stochastic function:

$$E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = E_{\mathbf{x}}\left[\ln P(y|\mathbf{x},\mathbf{w}) - \frac{\lambda}{2}||\mathbf{w}||_2^2\right]$$

- Batch gradient ascent updates:

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \frac{1}{N}\sum_{j=1}^{N} x_i^{(j)} [y^{(j)} - P(Y=1|\mathbf{x}^{(j)},\mathbf{w}^{(t)})] \right\}}$$

- Stochastic gradient ascent updates:
  - Online setting:

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta_t \left\{ -\lambda w_i^{(t)} + x_i^{(t)} [y^{(t)} - P(Y=1|\mathbf{x}^{(t)},\mathbf{w}^{(t)})] \right\}}$$

Convergence Rate of SGD

- **Theorem**:
  - (see Nemirovski et al '09 from readings)
  - Let *f* be a strongly convex stochastic function
  - Assume gradient of *f* is Lipschitz continuous and bounded
  - Then, for step sizes:
  - The expected loss decreases as O(1/t):

Convergence Rates for Gradient Descent/Ascent vs. SGD

- Number of Iterations to get to accuracy

$$\ell(\mathbf{w}^*) - \ell(\mathbf{w}) \leq \epsilon$$

- Gradient descent:
  - If func is strongly convex: O(ln(1/ϵ)) iterations
- Stochastic gradient descent:
  - If func is strongly convex: O(1/ϵ) iterations
- Seems exponentially worse, but much more subtle:
  - Total running time, e.g., for logistic regression:
    - Gradient descent:
    - SGD:
    - SGD can win when we have a lot of data
  - See readings for more details

What you should know about Logistic Regression (LR) and Click Prediction

- Click prediction problem:
  - Estimate probability of clicking
  - Can be modeled as logistic regression
- Logistic regression model: Linear model
- Gradient ascent to optimize conditional likelihood
- Overfitting + regularization
- Regularized optimization
  - Convergence rates and stopping criterion
- Stochastic gradient ascent for large/streaming data
  - Convergence rates of SGD