An example Fourier calculation

Proposition 1 *Suppose f is a boolean-valued function, $f : \{-1,1\}^n \to \{-1,1\}$, and suppose we do the following "test". We pick $x \in \{-1,1\}^n$ uniformly at random. Then we pick a random index J uniformly from $[n] := \{1, 2, \ldots, n\}$. Then we let $y \in \{-1,1\}^n$ be the string given by taking x and flipping (reversing) its Jth bit. E.g., if $n = 5$, we might pick x to be $(-1,1,1,-1,1)$, then we might pick $J = 4$, and so then $y = (-1,1,1,1,1)$.*

Then

$$\mathop{\mathbf{Pr}}_{x,J}[f(x) = f(y)] = \frac{1}{2} + \frac{1}{2}\sum_{S \subseteq [n]} (1 - 2|S|/n)^{|S|} \hat{f}(S)^2.$$

Proof: Since $f(x)$ and $f(y)$ have values in $\{-1,1\}$, we can write a 0-1 indicator for the event $f(x) = f(y)$ as follows:

$$\mathbf{1}[f(x) = f(y)] = \frac{1}{2} + \frac{1}{2} f(x) f(y).$$

Therefore

$$\mathop{\mathbf{Pr}}_{x,J}[f(x) = f(y)] = \mathop{\mathbf{E}}_{x,J}\left[\frac{1}{2} + \frac{1}{2} f(x) f(y)\right] = \frac{1}{2} + \frac{1}{2}\mathop{\mathbf{E}}_{x,J}[f(x) f(y)],$$

where in the second step we used linearity of expectation. Thus to show the proposition, it remains to show

$$\mathop{\mathbf{E}}_{x,J}[f(x) f(y)] = \sum_{S \subseteq [n]} (1 - 2|S|/n)^{|S|} \hat{f}(S)^2. \tag{1}$$

We now express $f(x)$ and $f(y)$ using the Fourier expansion, writing (1) as:

$$\mathop{\mathbf{E}}_{x,J}[f(x) f(y)] = \mathop{\mathbf{E}}_{x,J}\left[\left(\sum_{S \subseteq [n]} \hat{f}(S)\chi_S(x)\right) \times \left(\sum_{T \subseteq [n]} \hat{f}(S)\chi_S(y)\right)\right];$$

here we used different indices S and T for clarity. (Recall that when $S \subseteq [n]$, the function $\chi_S : \{-1,1\}^n \to \{-1,1\}$ is the ± 1-value parity function, $\chi_S(x) = \prod_{i \in S} x_i$.) Inside the expectation we have the product of two sums, each over 2^n terms; let's expand this out to a sum over $(2^n)^2$ products. We get

$$\mathop{\mathbf{E}}_{x,J}\left[\sum_{S \subseteq [n]} \sum_{T \subseteq [n]} \hat{f}(S)\chi_S(x)\hat{f}(T)\chi_T(x)\right].$$

We use linearity of expectation again to bring the sums to the outside and the expectation to the inside. Note that we can bring the quantities $\hat{f}(S)\hat{f}(T)$ outside of the expectation, since these are numbers just depending on S and T and not on x and i. We get:

$$\sum_{S \subseteq [n]} \sum_{T \subseteq [n]} \hat{f}(S)\hat{f}(T) \mathop{\mathbf{E}}_{x,J}\left[\chi_S(x)\chi_T(y)\right]. \tag{2}$$

Let's give a name to the quantities in the expectations. Specifically, given S and T, let's write

$$c_{S,T} = \mathop{\mathbf{E}}_{x,J} \left[\chi_S(x)\chi_T(y)\right].$$

Claim 2

$$c_{S,T} = \begin{cases} 0 & \text{if } S \neq T, \\ (1 - 2|S|/n)^{|S|} & \text{if } S = T. \end{cases}$$

Note that if we can prove this claim then we are done. That's because the claim implies that (2) is equal to

$$\sum_{S \subseteq [n]} (1 - 2|S|/n)^{|S|} \hat{f}(S)^2,$$

which is just what we need to prove to verify (1).

Let's prove the claim. First we expand out the definition of χ_S and χ_T:

$$\begin{aligned} c_{S,T} = \mathop{\mathbf{E}}_{x,J} \left[\chi_S(x)\chi_T(y)\right] &= \mathop{\mathbf{E}}_{x,J} \left[\left(\prod_{i \in S} x_i \right) \times \left(\prod_{i \in T} y_i \right) \right] \\ &= \mathop{\mathbf{E}}_{x,J} \left[\prod_{i=1}^{n} (x_i^{\mathbf{1}[i \in S]} \cdot y_i^{\mathbf{1}[i \in T]}) \right]. \end{aligned} \tag{3}$$

In the last step here we are letting $\mathbf{1}[i \in S]$ stand for the number 1 if $i \in S$ and the number 0 if $i \notin S$. (This is not a random variable as it was in our earlier use of the $\mathbf{1}$ notation; it's just a number.) It is perhaps a bit of a strange way to express things, but you should be able to easily agree that we have equality here.

What we have done so far is the "standard formula" for doing these Fourier-style calculations. Problems 6b), 6d), and 7c) all follow this recipe. The variation comes in how we now proceed. The remainder of this proof is actually harder than the remainder of the homework problems. Hint for the homework problems: Use the fact that the *pairs* or *triples* of random variables $(x_i, y_i, (z_i))$ are independent across the values of $i = 1 \ldots n$. (I.e., although x_i and y_i are not independent in the homework problems, (x_i, y_i) is independent of (x_j, y_j) for $i \neq j$.) Hence one can rewrite the expectation of the product across i as the product across i of expectations.

Back to our problem. Suppose that $S \neq T$. We want to show that (3) is 0 in this case. Since $S \neq T$, there must exists some index $\ell \in [n]$ which is in S but not T, or vice versa. (I.e., such that $\mathbf{1}[\ell \in S] \neq \mathbf{1}[\ell \in T]$.) Let's say that that ℓ is in T but not in S. (The opposite case is essentially identical.) In this case we can write (3) as

$$\mathop{\mathbf{E}}_{x,J} \left[y_\ell \cdot \prod_{\substack{i=1 \\ i \neq \ell}}^{n} (x_i^{\mathbf{1}[i \in S]} \cdot y_i^{\mathbf{1}[i \in T]}) \right].$$

Now we come to the subtlest point in the proof: We claim that the two quantities inside the expectation here, y_ℓ and $\prod_{i \neq \ell} (x_i^{\mathbf{1}[i \in S]} \cdot y_i^{\mathbf{1}[i \in T]})$ are independent random variables. This is because even if I tell you everything that goes into the second quantity — i.e., both J and all bits of x except for x_ℓ — you still learn nothing about y_ℓ. (This requires some reflection.) So since we have independence, we get

$$\mathop{\mathbf{E}}_{x,J}\left[y_\ell \cdot \prod_{\substack{i=1\\ i\neq\ell}}^{n}(x_i^{\mathbf{1}[i\in S]} \cdot y_i^{\mathbf{1}[i\in T]})\right] = \mathop{\mathbf{E}}_{x,J}[y_\ell] \times \mathop{\mathbf{E}}_{x,J}\left[\prod_{\substack{i=1\\ i\neq\ell}}^{n}(x_i^{\mathbf{1}[i\in S]} \cdot y_i^{\mathbf{1}[i\in T]})\right].$$

But $\mathbf{E}_{x,J}[y_\ell] = 0$, because one can see that any particular y_ℓ is equally likely to be -1 or 1. This completes half of the proof of the claim; we've shown that $c_{S,T} = 0$ when $S \neq T$.

For the second half of the claim, suppose $S = T$. Then (3) is equal to

$$\mathop{\mathbf{E}}_{x,J}\left[\prod_{i\in S} x_i y_i\right] = \mathop{\mathbf{E}}_{J}\left[\mathop{\mathbf{E}}_{x}\left[\prod_{i\in S} x_i y_i\right]\right] = \mathop{\mathbf{E}}_{J}\left[\prod_{i\in S}\mathop{\mathbf{E}}_{x}[x_i y_i]\right]. \tag{4}$$

The second equality here used the observation from the hint, above: once J is fixed, the pairs (x_i, y_i) are independent across i's. In particular, if $i = J$ then (x_i, y_i) is a random pair of opposite bits, and if $i \neq J$ then (x_i, y_i) is a random pair of identical bits. Hence we immediately get that

$$\mathop{\mathbf{E}}_{x}[x_i y_i] = \begin{cases} -1 & \text{if } i = J, \\ 1 & \text{if } i \neq J, \end{cases}$$

and therefore we conclude

$$\prod_{i\in S}\mathop{\mathbf{E}}_{x}[x_i y_i] = (-1)^{\mathbf{1}[J\in S]}.$$

Hence (4) is equal to

$$\mathop{\mathbf{E}}_{J}\left[(-1)^{\mathbf{1}[J\in S]}\right]. \tag{5}$$

This quantity clearly only depends on the size of S. In particular, the probability that $J \in S$ is $|S|/n$. So with probability $|S|/n$ we count -1 and with probability $1 - |S|/n$ we count 1. Thus (5) is equal to $1 - 2|S|/n$, and we have shown the second half of the claim, $c_{S,S} = 1 - 2|S|/n$. □