Problems:

1. Show that if $\mathsf{NP} \subseteq \mathsf{TIME}(n \log n)$ then $\mathsf{PH} \subseteq \bigcup_k \mathsf{TIME}(n \log^k n)$. $\Sigma^p_k \subseteq \mathsf{TIME}(n \log^{ck} n)$.

2. Show that $\mathsf{#2-SAT}$ is $\mathsf{#P}$-complete.

3. This problem will derive Lupanov’s bound on the worst-case size required to compute any Boolean function on $n$ bits. The key to this construction is to compute many functions of fewer than $n$ bits using a single circuit that has more than one node as an output node.

   (a) Show how to compute all conjunctions from $\{x_1, \ldots, x_m\}$ efficiently using a single circuit.

   (b) View the inputs to $f$ as defining a $2^k \times 2^{n-k}$ matrix. For some parameter $s \leq 2^k$ partition the rows of $f$ into groups of size $s$ and one remaining group. Then represent $f$ as $\bigvee_{i,v}(f_{i,v}(x_1, \ldots, x_k) \land f'_{i,v}(x_{k+1}, \ldots, x_n))$ for $i \in [p]$ and $v \in \{0, 1\}^s$ where $p = \lceil 2^k / s \rceil$, each of the functions $f_{i,v}$ is 1 on at most $s$ inputs, and for each $i \in [p]$ the functions $f'_{i,v}$ taken together are 1 on at most $2^{n-k}$ inputs.

   (c) Use the properties of part (b) to find an efficient construction using the circuit from (a) that computes $f$. Then choose values of $k$ and $s$ to optimize the construction and derive a size $2^n / n + o(2^n / n)$ circuit that computes $f$.

4. Even if $\mathsf{P} = \mathsf{NP}$ we do not know whether $\mathsf{#P} \subseteq \mathsf{FP}$.

   (a) Show that if $\mathsf{P} = \mathsf{NP}$ then for every $f \in \mathsf{#P}$ there is a randomized algorithm that approximates $f$ within a factor of 2. Hint: Use a hashing-based method for estimating the size of a set, as given in the proof of the Valiant-Vazirani lemma or Lautemann’s Lemma.

   (b) Under the same assumption, improve the approximation factor.