1. (15 points) In class, we have discussed various undecidability results relating to Turing machines such as Rice’s Theorem that deal with the nature of language they recognize. In this problem, we consider questions concerning the actual runtime behavior of Turing machines. Which of the following problems about Turing machines are decidable and which are not? Briefly justify your answers.

(a) To determine, given a Turing machine $M$, a string $w$, and integers $a$ and $b$ does $M$ run for more than $a|w|^2 + b$ steps on input $w$?

(b) To determine, given a Turing machine $M$, and integers $a$ and $b$ does there exist a string $w$ such that $M$ runs for more than $a|w|^2 + b$ steps on input $w$?

(c) To determine, given a Turing machine $M$ does $M$ have a useless state, a state that is never entered on on any input string.

2. (10 points) Define

$$\text{REGULAR}_{\text{CFG}} = \{ \langle G \rangle \mid G \text{ is a context-free grammar and } L(G) \text{ is regular} \}.$$ 

Prove that $\text{REGULAR}_{\text{CFG}}$ is undecidable. Ideas similar to those used to prove that $\text{ALL}_{\text{CFG}}$ is undecidable may be useful.


4. (10 points) A language $L$ is said to be $\leq_m$-complete for a class of languages $\mathcal{C}$ if and only if

- $L \in \mathcal{C}$ and
- For all languages $K \in \mathcal{C}$, $K \leq_m L$.

Prove that $A_{\text{TM}}$ and $\text{HALT}_{\text{TM}}$ are both $\leq_m$-complete for the class of Turing-recognizable languages.

5. (15 points) We say that a relation $R \subseteq \{0,1\}^k$ is decidable iff the language

$$L_R = \{ (x_1, x_2, \ldots, x_k) \mid (x_1, x_2, \ldots, x_k) \in R \}$$

is decidable.

Define $\Sigma_k$, for $k \geq 0$, to be the class of all languages $L$ for which there is a decidable $(k+1)$-ary relation $R$ such that

$$L = \{ x \mid \exists x_1 \forall x_2 \cdots Q_k x_k R(x_1, x_2, \ldots, x_k, x) \},$$
where the quantifier $Q_k$ is $\exists$ if $k$ is odd and $\forall$ if $k$ is even.

Define $\Pi_k = \{ L \mid L \in \Sigma_k \}$, i.e. $\Pi_k$ is the set of all complements of languages in $\Sigma_k$.

Equivalently, $\Pi_k$ is the class of all languages $L$ for which there is a decidable $(k + 1)$-ary relation $R$ such that

$$L = \{ x \mid \forall x_1 \exists x_2 \cdots Q_k x_k R(x_1, x_2, \ldots, x_k, x) \},$$

where the quantifier $Q_k$ is $\forall$ if $k$ is odd and $\exists$ if $k$ is even.

In this notation, clearly $\Sigma_0$ and $\Pi_0$ equal the set of decidable languages. Moreover, by problem 2 on homework 1, $\Sigma_1$ consists of precisely the Turing-recognizable languages.

(a) Prove that the language $E_{TM} = \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset \}$

is $\leq_m$-complete for $\Pi_1$.

(b) Prove that the language $ALL_{TM} = \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \{0,1\}^* \}$

is in $\Pi_2$ but not in $\Sigma_1$ or $\Pi_1$.

(c) Find the smallest $k$ you can such that the language $EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are Turing machines and } L(M_1) = L(M_2) \}$

is in $\Sigma_k$ or $\Pi_k$ and justify your answer.

(d) (Extra credit) Prove that $EQ_{TM}$ is $\leq_m$-complete for the class you found in part (c).