CSE 531: Computability and Complexity — Autumn 2003

**Problem Set #3** — Instructor: Venkatesan Guruswami

Due on Wednesday, **November 12, 2003** in class.

---

**Instructions:** Same as Problem Set 1.

There are **EIGHT** problems, including **TWO** *optional* problems.

---

1. Problem 7.26, Sipser's book ($PUZZLE$ is NP-complete)

2. Assume P = NP. Describe a polynomial time algorithm that takes as input an undirected graph and finds a largest clique contained in that graph. (A clique is a subset of vertices every pair of which are connected by an edge in the graph.)

   - Note that the algorithm you are asked for computes a function, and NP contains only decision problems. So arguing that "finding the largest clique is in NP, and since we are assuming P = NP, we are done" is not a (correct) solution.

3. Define the language

   $$MAX2SAT = \{\langle \phi, k \rangle \mid \phi \text{ is a 2CNF Boolean formula and there exists an assignment that satisfies at least } k \text{ clauses of } \phi\}\ .$$

   Prove that MAX2SAT is NP-complete.

4. Problem 7.37, Sipser's book. (2SAT is in P)

5. The cat-and-mouse game is played by two players, "Cat" and "Mouse", on an arbitrary undirected graph. At a given point each player occupies a node of the graph. The players take turns moving to a node adjacent to the one that they currently occupy. A special node of the graph is called "Hole". Cat wins if the two players ever occupy the same node. Mouse wins if it reaches the Hole before the preceding happens. The game is a draw if the two players ever simultaneously reach positions they previously occupied. Let

   $$HAPPYCAT = \{\langle G, c, m, h \rangle \mid G, c, m, h, \text{ are respectively a graph, and positions of the Cat, Mouse, and Hole, such that Cat has a winning strategy, if Cat moves first}\}\ .$$

   Prove that $HAPPYCAT$ is in P.

   **Hint:** Problem is somewhat tricky, but admits a very elegant solution.

6. Define the Boolean function $\text{MAJORITY}_n : \{0,1\}^n \to \{0,1\}$ as:

   $$\text{MAJORITY}_n(x_1, x_2, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} x_i \geq n/2\ .$$

   Thus $\text{MAJORITY}_n$ returns the majority vote of its inputs. Show that $\text{MAJORITY}_n$ can be computed with $O(n)$ size Boolean circuits (with NOT gates and fan-in 2 AND and OR gates). (Hint: Divide and Conquer)

7. * **(Optional Problem)** A language $L$ is said to be unary if $L \subseteq \{0\}^*$. Show that if a unary language is NP-complete, then P = NP.

8. * **(Optional Problem)** A bipartite graph is a graph whose vertices can be partitioned into two disjoint parts each of which is an independent set. Formally a bipartite graph $H = (X, Y, E)$ has vertex set $X \cup Y$ for disjoint sets $X, Y$ and each edge in its edge set $E$ has one endpoint in $X$ and one in $Y$. A $k$-bipartite clique of $H$ is a pair of subsets $S \subseteq X$ and $T \subseteq Y$ with $|S| = |T| = k$ such that $(s, t) \in E$ for each $s \in S$ and $t \in T$ (informally all "cross-edges" exist between $S$ and $T$). Define the language

$$\text{BIPARTITE-CLIQUE} = \{\langle H, k\rangle \mid H \text{ is a bipartite graph that has a } k\text{-bipartite clique}\} .$$

Prove that BIPARTITE-CLIQUE is NP-complete.