

Random Walks & Markov Chains

2-SAT algorithm

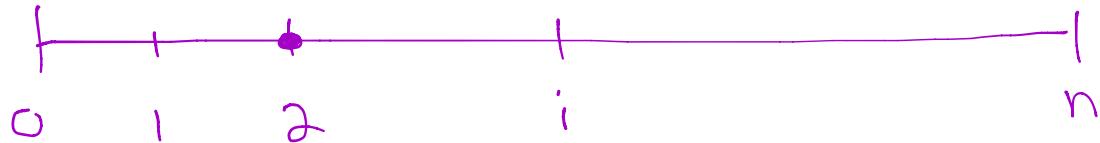
random walk based algorithm

- start w/ arbitrary assignment
- Repeat up to $100n^2$ times unless all clauses are satisfied
 - choose an arbitrary clause that is not satisfied
 - pick a var in that clause at random & switch its value
- Return satisfying assignment if found ; else return "unsatisfiable"

How to analyze?

Fix satisfying assignment S

Think of alg as random walk on line.



position = # vars on which current assignment
agrees with S

Let X_+ be position after t steps

from X_+ , move to either X_++1 or X_-1

Key claim:

$$\Pr(X_{t+1} = i+1 \mid X_t = i) \geq \frac{1}{2}$$

Alg succeeds if X_+ reaches n

Consider pessimistic version where

$$\Pr(X_{t+1} = i+1 \mid X_t = i) = \frac{1}{2}$$

$$\Pr(X_{t+1} = 1 \mid X_t = 0) = 1$$

standard random walk

Will show that $E(\min t \mid X_t = n) = n^2$

This implies expected # steps till actual alg succeeds

if formula satisfiable $\leq n^2$

$$\Rightarrow \Pr(\text{not finding satisfying assignment when one exists}) \leq \frac{1}{2^{100}}$$

h_j = expected # steps for random walk to reach n starting

at j

$$h_j = \frac{1}{2} h_{j-1} + \frac{1}{2} h_{j+1} + 1 \quad 0 < j < n$$

$$\Rightarrow h_j - h_{j+1} = h_{j-1} - h_j + 2$$

$$h_n = 0$$

$$h_0 - h_1 = 1$$

By induction, $h_j - h_{j+1} = 2j + 1$

$$h_0 = h_0 - h_n = \sum_{i=0}^{n-1} h_i - h_{i+1} = \sum_{i=0}^{n-1} (2i + 1) = 2 \frac{(n-1)n}{2} + n = n^2$$

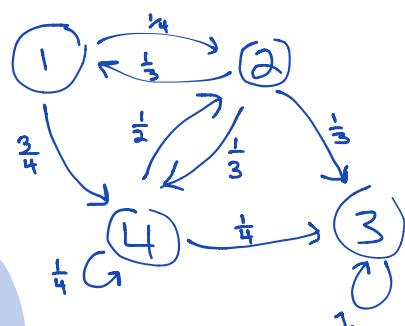
Finite Markov Chains

random walk on directed graph

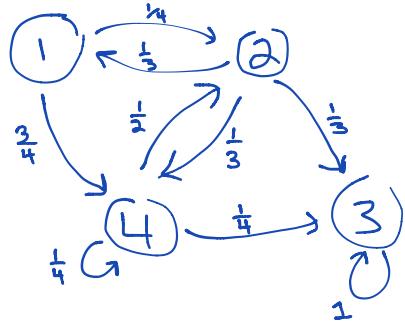
each vertex called "state" of M.C.

each arc describes corresponding

transition probability



Transition probabilities described w/ transition matrix $P = (P_{ij})_{\substack{i=1 \dots n \\ j=1 \dots n}}$



$$P = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 2 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Use X_t to denote state at time t .

$$\Pr(X_{t+1}=j | X_t=i) = P_{ij}$$

$\vec{p}^{(t)} = (p_1^{(t)}, \dots, p_n^{(t)})$ describes probability dist'n over states at time t

$$p_i^{(t)} = \Pr(X_t=i)$$

$\vec{p}^{(0)} = (1, 0, \dots, 0)$ means start in state 1

$\vec{p}^{(0)} = (\frac{1}{n}, \dots, \frac{1}{n})$ means start in uniformly random state.

Observe $\vec{p}^{(t+1)} = \vec{p}^{(t)} P$

$$\vec{p}^{(t+m)} = \vec{p}^{(t)} P^m$$

(called a Markov chain because it has "Markov property")

= next state depends on current state
but not on history

Irreducible Markov chain

corresponding graph strongly connected

$$\text{Period of state } i := \gcd \{n \geq 1 \mid p_{ii}^n > 0\}$$

Markov chain aperiodic if period of every state is 1

All Markov chains we will consider will be finite, irreducible
& aperiodic

$$\Rightarrow \exists N > 0 \text{ s.t. } P^n \text{ strictly positive } \forall n \geq N$$

A stationary dist'n $\vec{\pi}$ of a M.C. is a prob dist'n s.t.

$$\vec{\pi} = \vec{\pi} P \quad \text{"fixed pt"}$$

$$\forall j \quad \pi_j = \sum_i \pi_i p_{ij}$$

Fundamental Thm of Markov Chains

For any finite irreducible, aperiodic MC :

① \exists stationary distn $\vec{\pi}$

② $\vec{\pi}$ is unique

$$\textcircled{3} \quad \pi_i = \frac{1}{h_{i,i}}$$

$$\textcircled{4} \quad \forall i, j \quad \lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$$

hitting time

$$T_{ij} = \min_{t \geq 1} \{X_t = j \mid X_0 = i\}$$

$$h_{ij} = E(T_{ij})$$

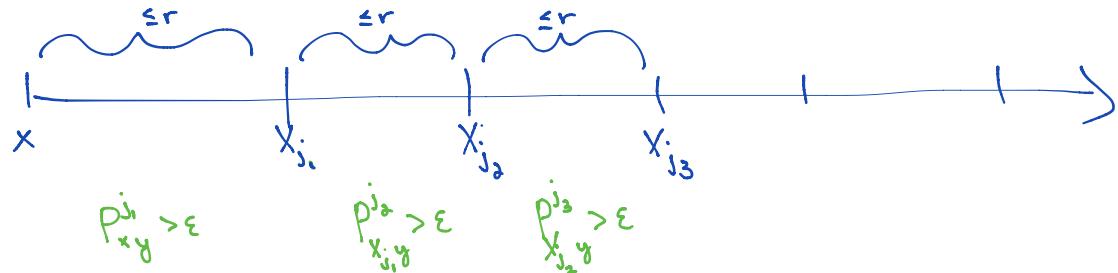
h_{ii} = expected first return time



Claim: $h_{xy} < \infty$ if chain irreducible

Pf:

$\forall z, w \quad \exists r, \varepsilon \text{ s.t. } P_{zw}^j > \varepsilon \quad (j \leq r)$



phases $\text{Geom}(\varepsilon)$

Proof of ① $\exists \pi$ s.t. $\pi P = \pi$ and $\pi_x > 0 \quad \forall x \in \mathcal{S}$

$$\begin{aligned} \text{Define } \tilde{\pi}_y &= E_z (\# \text{visits to } y \text{ before returning to } z) \\ &= \sum_{t=0}^{\infty} \Pr_z (X_t = y, \text{ still no } z) \\ &\leq h_{zz} < \infty \end{aligned}$$

Claim $\tilde{\pi}_y$ stationary

$$\begin{aligned} \sum_x \tilde{\pi}_x P_{xy} &= \sum_x \sum_{z^+} P_z (X_t = x; T_{zz}^+ > t) P_{xy} \\ &= \sum_t \sum_x P_z (X_t = x; T_{zz}^+ > t) P_{xy} \\ &= \sum_{t=0}^{\infty} \Pr_z (X_{t+1} = y; T_{zz}^+ \geq t+1) \end{aligned}$$

$$\begin{aligned} &= \sum_{t=1}^{\infty} \Pr_z (X_t = y; T_{zz}^+ > t) \\ &= \tilde{\pi}_y - \Pr_z (X_0 = y, T_{zz}^+ > 0) + \sum_{t=1}^{\infty} \Pr_z (X_t = y, T_{zz}^+ = t) \\ &= \tilde{\pi}_y - \underbrace{\Pr_z (X_0 = y)}_{\text{either both 0 or 1}} + \Pr_z (X_{T_{zz}^+} = y) \end{aligned}$$

$$\Rightarrow \sum_x \tilde{\pi}_x p_{xy} = \tilde{\pi}_y$$

$\tilde{\pi}_y = E_z (\# \text{ visits to } y \text{ before returning to } z)$

normalize to get prob measure

$$\pi_x = \frac{\tilde{\pi}_x}{\sum_y \tilde{\pi}_y} = \frac{\tilde{\pi}_x}{h_{zz}}$$

② $\vec{\pi}$ unique

Suppose not. $\vec{\pi}$ & $\tilde{\vec{\pi}}$ distinct

$$\text{let } x \text{ minimize } \frac{\pi_x}{\tilde{\pi}_x} = c$$

$$\frac{\sum_y \tilde{\pi}_y p_{yx}}{\sum_y \pi_y p_{yx}} \geq \frac{\sum_y \tilde{\pi}_y p_{yx}}{\sum_y \tilde{\pi}_y p_{yx}} > c$$

$$\textcircled{3} \quad \pi_i = \frac{1}{h_{ii}}$$

$$\begin{aligned}\pi_x &= \frac{\tilde{\pi}_x}{\sum_y \tilde{\pi}_y} = \frac{\tilde{\pi}_x}{h_{22}} \\ \text{Uniqueness} &= \frac{E_x \left(\# \text{of visits to } x \text{ before returning to } x \right)}{E_x (H_x^+)}\end{aligned}$$

(4) $\forall x, y \quad \lim_{n \rightarrow \infty} P_{xy}^n = \pi_y$

we know $\exists r \text{ s.t. } P^r > 0$

for $\delta > 0$ sufficiently small

$$P_{xy}^r \geq \delta \pi_y \quad \forall x, y$$

$$\mathbb{II} = \begin{pmatrix} \longleftrightarrow \pi & & \\ & \longleftrightarrow \pi & \\ & & \longleftrightarrow \pi \end{pmatrix}$$

$$P = \underbrace{\delta}_{1 - (1-\delta)} \mathbb{II} + (1-\delta) Q \quad \text{defines stochastic } Q$$

$$M \underline{\Pi} = \underline{\Pi} \quad \forall M \text{ stochastic} \quad \underline{\Pi} M = \underline{\Pi} \quad \text{if } \Pi M = \Pi$$

Claim: $P^{rk} = (1 - (1-\delta)^k) \underline{\Pi} + (1-\delta)^k Q^k$

proof by induction on k

$$\Rightarrow P^{rk+j} = [(1 - (1-\delta)^k) \underline{\Pi} + (1-\delta)^k Q^k] P^j$$

$$P^{rk+j} - \underline{\Pi} = (1-\delta)^k [Q^k P^j - \underline{\Pi}]$$

Maximum Matching in Regular Bipartite Graphs

By Hall's marriage thm, regular bipartite graphs always have perfect matching

random walk based alg that finds max matching in $O(n \log n)$ steps.

Traditional approach: augmenting path alg

repeatedly find one : can be done in $O(m)$ steps using BFS

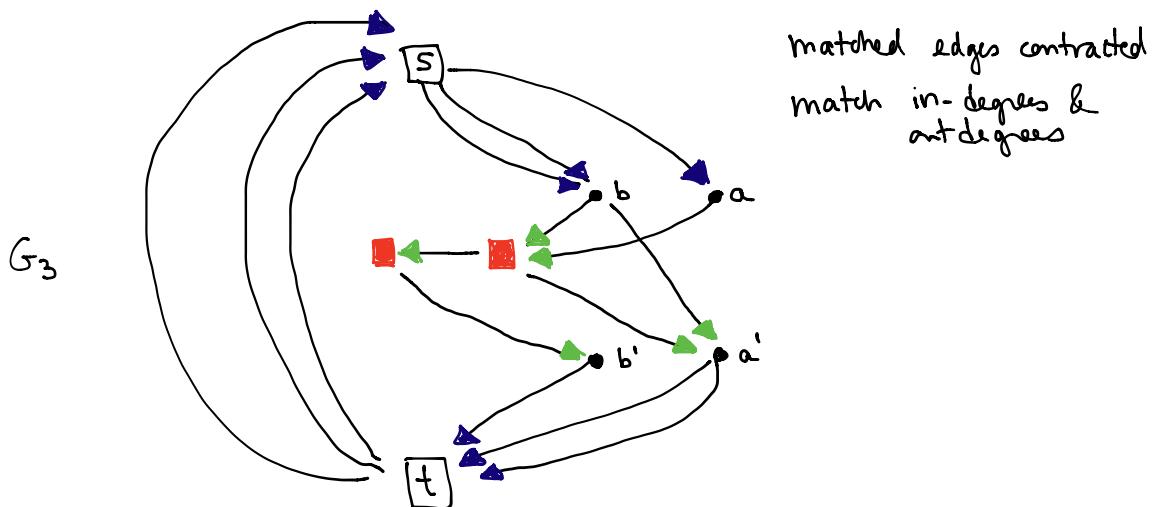
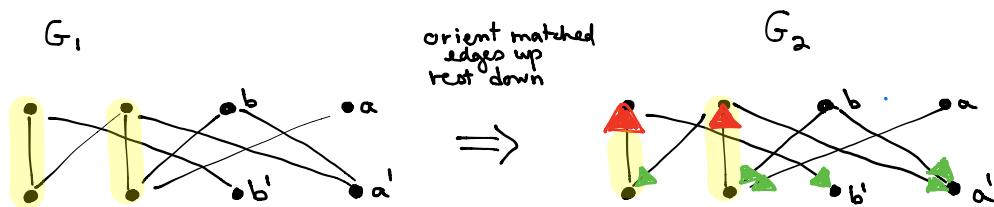
Theorem

A matching M is max iff no augmenting path

Pf AP \Rightarrow use to make bigger matching

$|M'| > |M|$ look at union of $M \& M'$

New idea: replace BFS with random walk!



Observations:

- ① G_3 is Eulerian - indegree of each node = outdegree

(2) G_1 has augmenting path iff

G_2 has directed path from top unmatched vertex to bottom unmatched vertex

iff G_3 has cycle from s to s

Alg to find augmenting path: random walk starting from s

Exp time to find augmenting path = $E(T_{s,s}) = h_{s,s}$

Recall

$$h_{s,s} = \frac{1}{\pi_s}$$

Claim:

In Eulerian directed graph, the stationary distn is

$$\pi_v = \frac{d^{\text{out}}(v)}{m} \quad \leftarrow \# \text{edges}$$

Proof:

$$\sum_{u|(u,v) \in E} \pi_u p_{uv} = \sum_{u|(u,v) \in E} \frac{d^{\text{out}}(u)}{m} \cdot \frac{1}{d^{\text{out}}(u)} = \frac{d^{\text{in}}(v)}{m} = \pi_v$$

$$\Rightarrow h_{s,s} = \frac{m}{d^{\text{out}}(s)}$$

In i^{th} iteration (i edges in matching) $d^{\text{out}}(s) \geq (n-i)d$

in d -regular graph.

$$\Rightarrow \text{in } i^{th} \text{ iteration } h_{s,s} \leq \frac{dn}{d(n-i)} = \frac{n}{n-i}$$

$$\Rightarrow \text{total running time } \sum_{i=0}^{n-1} \frac{n}{n-i} = O(n \log n)$$

$$\vec{\mu}^{t+1} = \vec{\mu}^t P$$

multiply distn by P
advances distn by
one step

invariant = stationary
 $\pi = \pi P$

$$\vec{h} = P \vec{g}$$

$$h(x) = \sum_y P_{xy} g(y)$$

multiply P by fn on states
gives exp of fn on states

invariant = harmonic
 $h(x) = \sum_y P_{xy} h(y) \equiv$ harmonic
at x

(harmonic everywhere
 \Rightarrow constant)