\[ p_{xy} = \frac{c_{xy}}{c_x} \quad T_x = \frac{c_x}{c_G} \quad G \text{ connected} \]

\[ L(G) = \begin{pmatrix} c_1 & 0 \\ c_2 & c_n \end{pmatrix} - \begin{pmatrix} i \\ -c_{ij} \end{pmatrix} \]

\[ = \sum_{e} c_e \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u \\ -1 \end{pmatrix} \quad e = (u, v) \]

\[ x^T L(G)x = \sum_{e=(u,v)} c_e (x_u - x_v)^2 \]

Laplacian p.s.d. \( \implies \lambda_i \geq 0 \)

\[ 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \quad \forall \lambda_i \text{'s are non-negative} \]

If \( G \) connected, \( \lambda_2 > 0 \)
Laplacian quadratic form highlights connection between Laplacian and cut structure.

Take $\mathbf{x}$ so $x_i = 1$ if $i \in S$, $0$ otherwise.

Then $\mathbf{x}^T \bar{L} \mathbf{x} = \sum_{e \in \delta(S)} c_e$, which measures capacity of cut $\delta(S)$.

No small cuts $\Rightarrow$ "expansion" $\Rightarrow$ rapid mixing $\Rightarrow$ spectral gap.

$\hat{\lambda}_2$, 2nd smallest eigenvalue of normalized Laplacian:

$$\hat{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}$$

$T'(\varepsilon)$ steps for lazy r.w. to converge to within $\varepsilon$ of $\Pi$

$$|\pi^0_i \frac{\tau'_{T'(\varepsilon)}}{\tau_{T'(\varepsilon)}} - \Pi_j| \leq \varepsilon \quad \forall j$$

$$T'(\varepsilon) = \frac{1}{\hat{\lambda}_2 \log \frac{n}{\varepsilon}}$$

$$C = \frac{\max_{c_e} c_e}{\min_{c_e} c_e}$$
Connections to cut structure

\[ \Gamma_G = \text{conductance} = \min_{S \neq \emptyset} \Phi(S) \]

\[ \Phi(S) = \frac{\sum_{e \in \partial S} c_e}{\min \left( \sum_{e \in \partial \text{in}} c_e, \sum_{e \in \partial \text{out}} c_e \right)} \]

Cheeger's Inequality:

\[ 2 \sqrt{\Gamma_G} \geq \lambda_2 \geq \frac{\Gamma_G^2}{2} \]

cuts of small conductance are obstacles to rapid mixing
in some sense only obstacles
Back to electrical flows:

If electric flow $\vec{f}$, field vector $\phi$ by vertex potentials for unit s-t current, compute $\vec{f}$ by solving system

$$L_G \phi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$c_x \phi_x - \sum_{y \in N(x)} c_{xy} \phi_y = 0$$

$$c_s \phi_s - \sum_u c_{su} \phi_u = 1$$

$$\sum_u c_{su} (\phi_s - \phi_u) = \sum_u f_{su} = 1$$

$$\hat{\varepsilon}_r(f) = f^T R f = \phi^T L(G) \phi$$

\[ \begin{align*}
\sum_e Ce (\phi_e - f_e)^2 &= \sum_e Ce \frac{f_e^2}{2e} = \sum_e \frac{Ce}{e} f_e^2
\end{align*} \]
Applications

Spectral Spanification

Given $G = (V, E, c)$ find $H = (V, F, w)$

sparse $|F| = O(n)$

s.t. $H$ approximates $G$ well

$\equiv$ all cuts approximately preserved $++$

$X^T L_H X \approx X^T L_G X$ for $X$ (eigenvalues $\approx$ same)

Example: degree $d$ expanders approx complete graph

Sample edges w/ prob $\propto$ effective resistance between ends.
Solving symmetric diagonally dominant linear systems

\[ Ax = b \quad A \text{ symm} \quad A_{ii} \gg \sum_{j \neq i} |A_{ij}| \]

[Spielman Tang] \ldots [Kelner, Orecchia, Sidford, Zhu]

Numerous applications: scientific computing, machine learning, vision, ...
+ algorithmic applications - flow, sparsification, routing, etc.

\[ \text{reduces} \rightarrow \nabla f = f_{s,t} \quad \equiv \text{electrical flow problem} \]

New alg: start with unit s-t flow (e.g. \( s-t \) path)

If it was electrical flow \( \nabla f \) cycle \( C \)
\[ \sum_{e \in C} \nabla f_e = 0 \]

- randomly sample cycle from well-chosen dist\( h \)

- compute \( \sum_{e \in C} \nabla f_e \) \& if \( \neq 0 \), add flow around cycle to make this \( 0 \).

Show that w\( h \) suitable data structures can be implemented efficiently in not too many cycle updates
Max Flow [Christiano et al., Srivastava/Rao]

- Compute electrical flow by solving $L \cdot f = \delta$ s.t. efficiency capacities will be violated

**Version 1**

Penalize congested edges by $1 \cdot r_e$

repeat

**Version 2**

$(u \rightarrow v)$ overly congested

$=>$ create current see at $v$ if value $= \text{excess flow}$

remove current at $u$

repeat
Random Spanning Trees

Procedure: Fix ordering on vertices \( v_1, \ldots, v_n \)

Take \textit{rw} starting at \( v_2 \) till hit \( r \), erasing any cycles.

Add edges on path to tree

If \( T \) spanning stop

If not, take first vertex in ordering that isn’t visited

Take \( r \cdot w \) erasing cycles until hit tree

Repeat... till tree is spanning

This tree is uniformly random tree!!! [Wilson]
Claim: Suppose fix a root \( r \) & select oriented tree

\[
\text{edge directed towards root with prob } \propto \prod_{e \in T} p_e^2
\]

Then if orientation & root are forgotten

\[
Pr(T) \propto \prod_{e \in T} c_e
\]

\[
\text{fix } r: \Pr(T^r) \propto \prod_{e \in T} p_e^2 = \frac{\prod_{e \in T} c_e}{\prod_{v \in V \setminus r} c_v}
\]

\[
\propto \frac{\prod_{e \in T} c_e}{\prod_{v \in V \setminus r} c_v} \quad \text{since } c_r \text{ same for all } T^r
\]

\[
\propto \prod_{e \in T} c_e \quad \text{since denom doesn't depend on } T
\]

Theorem

Wilson's algo finds tree in \( T^r \) w/ prob \( \propto \prod_{e \in T} p_e^2 \)

Therefore once root & orientations forgotten, find tree

w/ prob \( \propto \prod_{e \in T} c_e \)

Proof: Order in which cycles popped off stack doesn't matter
**Lemma**
\[ Pr((x,y)\in T) = c_{xy} \cdot R_{x\leftrightarrow y} \]

**Proof**
Run Wilson's alg with root y, starting at x

\[ Pr(e\in T) = Pr_x(\text{first hit y, walking along } e) \]

\[ = \sum_t Pr(\text{return to x at time } t \text{ without seeing } y) \cdot \frac{ce}{cx} \]

\[ = E_x(\# \text{ visits to x without seeing } y) \cdot \frac{ce}{cx} \]

\[ = cx \cdot R_{x\leftrightarrow y} \cdot \frac{ce}{cx} = ce \cdot R_{x\leftrightarrow y} \]

**Negative correlation:**
\[ Pr(e\in T \text{ and } f\in T) \leq Pr(e\in T) \cdot Pr(f\in T) \]

**Proof:**
\[ Pr(e\in T \land f\in T) = Pr(e\in T) \cdot Pr(f\in T | e\in T) \]

\[ = Pr(e\in T) \cdot Pr(f\in T | e) \]

\[ \text{amplitude of } e \text{- fused} \]
\[ \equiv \text{ resistance } 0 \]
These results used to get best known alg for TSP problems
Kirchoff’s Matrix Tree Thm

\[ \sum_{T} \prod_{e \in T} c_e = \frac{1}{n} \prod_{i=2}^{n} \lambda_i \]

Eigenvalues of Laplacian