**Puzzle.**

$n$ prisoners brought into room. An infinite pile of hats each independently & equally likely to be Red or Blue.

Without communicating, they must each write down the index of the first red hat on their head.

They are all shot unless they are all right.

Show that there is a strategy they can agree on ahead of time so that probably they survive is $\sum \left( \frac{1}{T \log n} \right)$.
Bipartite Matching

Given bipartite graph \((L, R, E)\),
a matching \(M \subseteq E\) is a set of edges that share no common endpoints.

Common goal: find matching of maximum size.

Online Bipartite Matching

\(L\) is known ahead of time.
Vertices in \(R\) arrive one at a time.
When \(j \in R\) arrives,
learn which vertices in \(L\) are neighbors of \(j\).
make an irrevocable decision as to which neighbor if any to match \(i\) to.

Killer app:
Internet ad allocation
match page views/ad slots with 'advertised' demand.
Greedy Alg: match arriving node to any neighbor
always obtains matching of size \( \geq \frac{1}{2} \text{OPT} \)

Analysis: every time Greedy adds an edge to the matching, think \( \$1 \) is earning \( \$1 \)
Place \( 50 \text{¢} \) on each endpoint.

Now consider any edge \((i;j)\) matched by OPT (when \( j \) arrives)
- if Greedy matches \( j \) then charge OPT's edge to \( 50 \text{¢} \) on \( j \)
- if Greedy can't match \( j \), then \( i \) is already matched
  charge OPT's edge to \( 50 \text{¢} \) on \( i \)

\( \Rightarrow \forall \$1 \) OPT earns, Greedy earns \( \geq 50 \text{¢} \).

The competitive ratio of Greedy
\[
\text{Greedy} = \max_I \frac{\text{size of Greedy matching on } I}{\text{size of OPT matching on } I}
\]
\[
= \frac{1}{2}
\]

No deterministic alg can do better.

So we turn to randomization...
But first, on easier problem: fractional version

When a node $j \in R$ arrives, can allocate it fractionally.

Claim: Let $A$ be a randomized alg for integral matching.

\exists deterministic fractional alg $D$ s.t. $\forall$ instance $I$

$$\sum_{i\in I} x_{ij}^D(I) = E(\sum_{i\in I} x_{ij}^A(I))$$

Let $D$ "simulate $A"

when $j$ arrives, set $x_{ij}^0 = \Pr(X_{ij}^A = 1)$

$$\forall i, \sum_{j \in N(i)} x_{ij} \leq 1 \implies E(\sum_{j \in N(i)} x_{ij}) \leq 1 \implies \sum_{j \in N(i)} x_{ij} \leq 1$$

$$\implies \sum_{(i,j) \in E} x_{ij} = E(\sum_{(i,j) \in E} x_{ij}) = E(\text{performance of } A)$$

$$\Rightarrow \text{upper bound on c.r. of deterministic fractional alg } \Rightarrow$$

upper bound on c.r. of randomized alg.

Upper bound on fractional c.r.
\[ i^{th} \text{ vertex gets } \frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{i} = H_n - H_{i-1} \]

\[ \ln \frac{n}{i} = 1 \]

\[ \text{when } i = \frac{n}{e} \]

matching has size \( n(1 - \frac{1}{e}) \)

OPT matching has size \( n \)

If don't divide equally worse

**Water Level alg:**

Imagine LHS water containers, capacity 1

RHS each vertex is a source of 1 unit of water

When vertex arrives, fill up neighbors in obvious way

1 unit water
Analysis:
again split gain between nodes, but when water level is high, want to leave relatively more money to protect against possibility of mistake

\[
g : [0,1] \rightarrow [0,1] \quad g(y) \uparrow
\]

- increase \( x_i \) by \( g(y_i) \) \( dx \)
- increase \( \beta_j \) by \( (1-g(y_j)) \) \( dx \)

where

\[
y_i = \sum_{j \in N(i)} x_{ij}
\]

\[
\sum_i \sum_j x_{ij} = \sum_i x_i + \sum_j \beta_j
\]

Want to choose \( g(\cdot) \) to prove best possible c.f.

Proof outline: Suppose OPT assigns \( x_{ij} \) to edge \((i,j)\)

Want to choose \( g \) so that no matter what:
\[(\alpha_i + p_j) x_{ij} > c x_{ij}^{\ast} \quad \text{(\star)}\]

for largest possible value of \(c\)

if so,

\[
\sum_{i} \alpha_i + \sum_{j} p_j \geq \sum_{i} \alpha_i \sum_{j \in W(i)} x_{ij} + \sum_{j} p_j \sum_{j \in W(j)} x_{ij}^{\ast}
\]

\[= \sum_{(i,j) \in E} (\alpha_i + p_j) x_{ij}^{\ast} \geq \sum_{(i,j) \in E} c x_{ij}^{\ast} \geq c \text{OPT} \quad \text{(\star)}
\]

Fix edge \((i,j)\)

Let \(y_{i,j}^{f}\) be final water level in node \(i\)

Case 1: \(y_{i,j}^{f} = 1\)

\(x_{ij}\) could even be 0, in which case \(p_{ij} = 0\)

\[\Rightarrow \alpha_i = \int_{0}^{1} g(x) dx = G(1) - G(0) \quad \text{where} \quad G'(x) = g(x)
\]

\[\Rightarrow \text{for c.r. c need} \quad G(1) - G(0) = c
\]

Case 2: \(y_{i,j}^{f} < 1 \Rightarrow j \text{ fully matched} \Rightarrow \sum_{i \in W(j)} x_{ij} = 1
\]

Claim \(\beta_i \geq 1 - g(y) > 1 - g(y_i)\)

\[\alpha_i + p_j \geq G(y_{i,j}^{f}) - G(0) + 1 - g(y_{i,j}^{f})
\]
want \( G(y) - G(0) + 1 - g(y) \geq c \quad \forall y \in [0, 1] \)

\[ g(y) - g'(y) = 0 \implies g(y) = ke^y \]

Substituting back in

\[
\int_0^y ke^x \, dx + 1 - ke^y = c
\]

\[
= ke^y - ke^0 + 1 - ke^y = 1 - k
\]

to find \( k \), use \( G(1) - G(0) = c \)

\[
= \int_0^1 ke^x \, dx = k(e-1)
\]

Combining we get

\[
1 - k = k(e-1) \implies k = \frac{1}{e}
\]

\[
C = 1 - \frac{1}{e}
\]

optimal!
Equivalent to primal-dual analysis

Primal
\[
\begin{align*}
\max & \sum_{(i,j) \in E} x_{ij} \\
\text{s.t.} & \sum_{j \in N(i)} x_{ij} \leq 1 \quad \forall i \quad \alpha_i \\
& \sum_{i \in N(j)} x_{ij} \leq 1 \quad \forall j \quad \beta_j \\
& x_{ij} \geq 0 \quad \forall (i,j) \in E
\end{align*}
\]

Weak duality \(\Rightarrow\) feasible primal \(\leq\) \(\text{OPT} \leq\) feasible dual

Dual
\[
\begin{align*}
\min & \sum_{i} \alpha_i + \sum_{j} \beta_j \\
\text{s.t.} & \alpha_i + \beta_j \geq 1 \quad \forall (i,j) \in E \\
& \alpha_i \geq 0 \quad \forall i \\
& \beta_j \geq 0 \quad \forall j
\end{align*}
\]

We constructed near-feasible dual, by ensuring

\[ \forall (i,j) \in E \quad \alpha_i + \beta_j \geq 1 - \frac{1}{2} \]

\[ \Rightarrow \hat{\alpha}_i = \frac{\alpha_i}{1 - \frac{1}{2}} \quad \hat{\beta}_j = \frac{\beta_j}{1 - \frac{1}{2}} \quad \text{feasible dual} \]

\[ \Rightarrow \text{OPT} \leq \sum_{i} \hat{\alpha}_i + \sum_{j} \hat{\beta}_j = \frac{1}{1 - \frac{1}{2}} \left[ \sum_{i} \alpha_i + \sum_{j} \beta_j \right] = \frac{1}{1 - \frac{1}{2}} \sum_{(i,j) \in E} x_{ij} \]
Integral online matching

**Ranking:** Select random total ordering $T$ of els $y_L$ when new vertex in $R$ arrives, match it to highest ranked neighbor according to $T$.

[Karp, Vazirani, Vazirani] 1990

[Goel, Mehta] 2008

[Birnbaum, Mathieu] 2008

New version: [Devanur, Jain, Kleinberg] 2013

First idea: online fractional matching $1 - \frac{1}{e}$

Every fractional matching is convex comb of integer solutions
Specifically, construct

Online randomized rounding that constructs integral matching
whose exp size = value of fractional matching produced by

Water Level.

\[\text{with prob } \frac{1}{2} \quad 1 \& 2\]

go wrong way

so exp value of integral

matching <

value of WL
Analysis:
Reinterpretation of alg:

\[ \forall i \in L \text{ pick } y_i \in \{0,1\} \text{ u.a.r.} \]

when \( j \in R \) arrives, match it to \( \arg\min \{ y_i : i \in N(j), i \text{ unmatched} \} \)

Prove very similar to analysis we just saw!

1. build feasible primal soln \( P = \{ x_i \mid (i,j) \in E, x_j \in \{0,1\} \} \)
2. (not necessarily feasible) randomized dual
   \( D = \{ \alpha_i, \beta_j \mid i \in L, j \in R \} \)
3. such that
   \( P \geq cD \quad \text{always} \)
   \( E(D) \) feasible

If so, \( E(P) \geq c\ OPT \)

Proof:
\[ P \geq cD \implies E(P) \geq c \bigg( E(D) \bigg) \geq c\ OPT \]

feasibility of \( E(D) \)
To construct duals, use \( g: [0,1] \rightarrow [0,1] \) \( g \uparrow \quad g(1) = 1 \)
when \( x_{ij} := 1 \) \set \alpha_i := \frac{g(y_i)}{c} \quad \beta_j := \frac{1 - g(y_i)}{c} \)

\[ \forall \text{unmatched } i,j \set \alpha_i := 0, \beta_j := 0 \]

\[ \Rightarrow j \text{ gets matched to } i \quad \Delta p = 1 \quad \Delta d = \frac{1}{c} \]

Just need to show \( E(0) \) is feasible.

Consider edge \((i,j)\) \quad Fix \( y_{-i} \)

Consider execution on \( G_{-i} \) \quad let \( \bar{y} = \begin{cases} 0 & \text{j not matched} \\ y_i & \text{j matched to i} \end{cases} \)

\( \beta_j = \frac{1 - g(y_i)}{c} \)

\[ \begin{align*}
\text{Lemma} \\
1. \text{In actual execution } \beta_j \geq \beta_j^* \\
2. \text{In actual execution, i matched if } y_i < y_i^* \\
\end{align*} \]

\[ \text{Lemma } \Rightarrow \forall y_{-i} \text{ fixed } \forall (i,j) \in E \quad E_{y_i}[\alpha_i + \beta_j] \geq 1 \]
\[ E_y [x_i] = \int_0^y \frac{g(y) dy}{c} \quad \beta^i_j = \frac{1 - g(y_i)}{c} \]

For \( g(y) = e^{-y} \) we know & have
\[ E_y (x_i + \beta^i_j) \geq G(y_i) - G(0) + \frac{1 - g(y_i)}{c} \geq 1 \]

with \( c = 1 - \frac{1}{e} \)

Proof of Lemma: In actual execution, \( i \) matched if \( Y_i < y_i \)
Proof of Lemma: In actual execution, $\beta_j \geq \hat{\beta}_j$

1. $j$ gets matched to $i$ with
   \[ y_j \leq y_i \]

2. \[ \Rightarrow \beta_j \geq \hat{\beta}_j \]