Multi-unit Auctions with Budget-constrained Bidders

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ABSTRACT

We study a multi-unit auction with multiple bidders, each of whom has a private valuation and a budget. The truthful mechanisms of such an auction are characterized, in the sense that, under standard assumptions, we prove that it is impossible to design a non-trivial truthful auction which allocates all units, while we provide the design of an asymptotically revenue-maximizing truthful mechanism which may allocate only some of the units. Our asymptotic parameter is a *budget dominance parameter* which measures the size of the budget of a single agent relative to the maximum revenue. We discuss the relevance of these results for the design of Internet ad auctions.

Categories and Subject Descriptors

J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*; F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity

General Terms

Algorithms, Economics, Theory

1. INTRODUCTION

Budget constraints are a central feature of many real auctions. In the context of e-commerce, there is a great deal of interest in multi-unit auctions of relatively low-value goods, such as the auction of Internet ads for search terms and content pages on MSN, Google, Yahoo, etc., to bidders with budget constraints. Indeed, it is widely believed that advertising will be the principal business model for online activity, and that budget-constrained auctions will be the primary means of realizing that revenue stream. Auctions with budget constraints have been considered previously in the context of privatization of high-value public goods, such as FCC auctions of telecommunications bands [2, 3, 10, 12]. However, the theoretical framework of budget-constrained auctions is currently substantially less well-developed than that of unconstrained auctions – which is unsatisfactory both from a theoretical viewpoint, and from a practical viewpoint, where the absence of an appropriate framework leads to losses in revenue and efficiency. It is therefore of tremendous interest to design an incentive-compatible allocation (i.e., truthful) mechanism for budget-constrained auctions, and indeed, to determine the circumstances under which such a mechanism even exists.

In this paper, we consider the problem of a multi-unit auction with multiple bidders, each of whom has a private valuation for each unit of the good and a budget. We prove both an impossibility result and a constructive, positive result. Throughout the paper, we assume the very natural conditions of enforcement of supply limits, individual rationality, and incentive compatibility (see Section 2 for definitions of these terms).

Existence of incentive-compatible mechanisms for budget-constrained bidders is a technically non-trivial problem. We assume that each bidder has a fixed private valuation and budget, such that if this budget is exceeded, then the bidder's total utility becomes unbounded below. We sometimes call this a "hard" budget constraint to distinguish it from "flexible" budget constraints considered by other authors [12], where the constraints can be exceeded under certain circumstances. Somewhat surprisingly, the well-known VCG mechanism is not truthful in the case with hard budgets – either with private hard budgets or in the *a priori* easier case with public hard budgets. This is an easy consequence of the fact that the utilities are not quasi-linear, and will be demonstrated explicitly in Section 2.

Section 4 contains the proof of our impossibility result. We show that, under the assumption that the auction sells all units, then in the two-unit, two-bidder case, the only deterministic incentive-compatible mechanism is a bundling mechanism. In other words, there is no truthful mechanism that sells the units to distinct buyers. Our proof follows from a tedious, but elementary analysis of the constraint

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equations, and uses characterization results proved in Section 3. We note that, while the condition of selling all units seems quite restrictive, it turns out that in this case, it is implied by the often assumed condition of "independence of irrelevant alternatives" (see [11]).

Section 5 contains the proof of our positive result. There we use randomization and relax the condition of selling all units allowing us to construct an incentive-compatible mechanism which asymptotically achieves revenue maximization. We introduce the *budget dominance parameter*, defined to be the maximum budget of any single bidder divided by the optimal, omniscient revenue. As in the work of Goldberg *et al.* [7, 8], we define the competitive ratio to be the ratio of the optimal revenue to the revenue of our mechanism. We then introduce an incentive-compatible mechanism for the general m-unit, n-bidder auction, and prove that, as the budget dominance parameter tends to zero, the competitive ratio tends to one.

Our mechanism is inspired by the work of Goldberg *et al.* [7, 8], but is different in several significant respects. First, our problem is a *two*-parameter problem, since each bidder specifies both a private valuation and a private budget. Second, our utility function is not quasi-linear. To our knowledge, there are no previously known incentive-compatible revenue-maximizing mechanisms in either of these cases. Finally, in the problem considered by [7, 8] at most one unit was allocated to each agent, whereas our setting has no such restriction. Due to these differences, our proof is substantially more complicated than that of [7, 8], requiring delicate martingale arguments to prove the necessary concentration result.

Let us also contrast our work with the some of previous work in the economics community which addressed budget constraints in a Bayesian setting [2, 3, 10, 12]. Che and Gale [2, 3] studied the single-item, single bidder case. Motivated by the goal of modelling efficient redistribution of public goods to the private sector, Maskin [12] studied the single-item, multiple bidder case with flexible budget constraints. The work closest to our context is that of Laffont and Robert [10], who treated a problem similar in some respects to the one studied here, but appropriate for a different set of applications. They too considered a two-parameter, non-quasi-linear, budget constrained problem. But they treated only single-item auctions with common public budgets. Moreover, whereas they examined the Bayesian equilibria, we consider *dominant strategy.* While their proposed mechanism, namely an all-pay auction, makes sense in the Bayesian context, it would not be appropriate as a dominant strategy, nor would it provide a reasonable mechanism in the case of online ad auctions. Other papers that consider the setting of auctions with budget constraints include [1, 5, 6, 15].

2. SETTING

We consider a setting in which an auctioneer has several indivisible units of a single good which he would like to auction off to n interested agents. Each agent i has a private utility $u_i \in \Re_+$ per unit of the good and a private budget constraint $b_i \in \Re_+$. We denote the vectors $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ and $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ by u_{-i} and b_{-i}, respectively. The budget constraint is a *hard* constraint, i.e., the agent cannot spend more than his budget under any circumstances. In other words, the total utility $u_i(j,p)$ that agent i derives from an allocation of j units at a total price of p is:

$$u_i(j,p) = \begin{cases} ju_i - p & \text{if } p \le b^i, \\ -\infty & \text{if } p > b^i. \end{cases}$$

The value $-\infty$ in the above definition means that this agent prefers receiving nothing and paying nothing to any lottery with a non-zero risk of going over the budget.

An auction mechanism solicits a two-parameter bid from each agent. The first parameter is interpreted as that agent's announced utility per unit and the second parameter is that agent's announced budget. The mechanism then outputs an allocation and payment for each agent. We consider mechanisms that satisfy the following properties:

- *observe supply limits* – The mechanism never allocates more units than are available.
- *individual rationality* – An agent's utility from participating in the mechanism is non-negative.
- *incentive compatibility* or *truthfulness* – An agent's total utility is maximized by announcing his true utility and budget to the auction regardless of the strategies of the other agents.

These properties can be generalized for randomized mechanisms by replacing *utility* by *expected utility.* In other words, we make the assumption that the agents are risk-neutral.[1] We call an auction mechanism satisfying the above properties a *truthful mechanism.* Notice that individual rationality and the definition of utility functions imply that a truthful mechanism never charges an agent an amount more than his budget.

One important distinction of the above setting compared to other models usually studied in the auction theory is that in our setting, the agents' utility functions are not *quasi-linear*. A quasi-linear utility function is a utility function of the form $u(x) - p$, where $u(x)$ is a function that only depends on the allocation and not on the payments, and p is the amount charged to the agent. The utility function $u_i(j,p)$ defined above cannot be written in this form. This makes many of the results in the auction theory literature inapplicable to our setting. In particular, the classical Vickrey-Clarke-Groves (VCG) mechanisms [14, 4, 9] are *not* incentive compatible in our setting. This fact is illustrated in the following example.

EXAMPLE 2.1. *One natural mechanism for auctioning m units of a good to budget-constrained buyers is to apply the VCG mechanism assuming that the utility of agent i for j units of the good is $\min(b_i, ju_i)$. A common mistake is to assume that since this mechanism is based on VCG, it is truthful. The following example shows that this is not the case: assume we have two units of the good to sell to two agents, and the truthful bids of these agents are given by $(u_1, b_1) = (10, 10)$ and $(u_2, b_2) = (1, 10)$. The above mechanism assumes that the utility of the first agent for either one or two units of the good is 10, and therefore allocates*

[1]The only place that we use this assumption in the algorithm given in Section 5 is in reduction from the indivisible problem to the divisible problem. Therefore, if the units of the good are divisible, the algorithm of Section 4 is strategyproof even if the agents are not risk-neutral.

one unit to each agent to maximize the total utility (which is $10+1$). The payment charged to the agents by this mechanism is 1 and 0, respectively. Therefore, the utility of the first agent is 9. However, if the first agent announces the bid $(5, 10)$, then the mechanism will allocate both units to this agent at a total price of 2. Thus, the first agent would achieve a utility of 18 by bidding untruthfully. This example shows that the above VCG-based mechanism is not truthful even if the agents are not allowed to lie about their budget.

It is easy to observe that in our setting, no truthful mechanism can always produce an *efficient* allocation, i.e., an allocation that maximizes the social welfare, even when there is only one good. The reason for this is that an efficient mechanism should always allocate the good to the bidder with the highest u_i, even if such a bidder has a zero budget and therefore cannot be charged any positive amount. Therefore, any agent can bid a high utility and zero budget to get the item for free. This simple impossibility result shows that we cannot require efficiency from a truthful mechanism.

3. CHARACTERIZATION

In this section, we give a simple characterization of truthful auctions. It essentially claims that any truthful auction determines the allocation and price for agent i by comparing his bid to thresholds computed from the other agents' bids. The unconstrained budgets version of this proposition is a well-known folklore theorem.

PROPOSITION 3.1. *For any deterministic truthful auction selling m units of a good to n agents, there exist mn functions $p_i^1, \dots, p_i^m : \Re_+^{2(n-1)} \to \Re_+ \cup \{\infty\}$ such that agent i receives j units at price $p_i^j(u_{-i}, b_{-i})$ where j maximizes $ju_i - p_i^j(u_{-i}, b_{-i})$ subject to $p_i^j(u_{-i}, b_{-i}) \le b_i$.*

PROOF. For any $(u_{-i}, b_{-i}) \in \Re_+^{2(n-1)}$ and $j \in \{1, \dots, m\}$, we define $p_i^j(u_{-i}, b_{-i})$ as the minimum, over the choice of (u_i, b_i) such that the auction allocates at least j units to i if agents bid (u, b), of the price that the mechanism charges to i at these bids. For any set of bids (u, b), let j^* be an index that maximizes $j^*u_i - p_i^{j^*}(u_{-i}, b_{-i})$ subject to $p_i^{j^*}(u_{-i}, b_{-i}) \le b_i$. If when agents bid (u, b), the mechanism allocates j units to i at price p, then we must have $ju_i - p = j^*u_i - p_i^{j^*}(u_{-i}, b_{-i})$, since otherwise agent i would have an incentive to bid untruthfully to get j^* units at price $p_i^{j^*}(u_{-i}, b_{-i})$. $\square$

By considering all cases for the relationship between the p_i^j's, the auction can be expressed as a concise set of inequalities. This is done for the case of two units of good and two buyers in the following corollary. We will use this corollary in the next section to prove that truthful mechanisms satisfying certain properties do not exist.

COROLLARY 3.1. *For any deterministic truthful auction selling 2 units of a good to 2 agents, there exist threshold functions $p_i^j : \Re_+^2 \to \Re_+ \cup \{\infty\}$, $1 \le i, j \le 2$, such that for $i = 1, 2$, the agent i receives*

- *2 units at a total price of $p_i^2(u_{3-i}, b_{3-i})$ if*
$$b_i \ge p_i^2(u_{3-i}, b_{3-i})$$
and
$$u_i > p_i^2(u_{3-i}, b_{3-i}) - \min(p_i^1(u_{3-i}, b_{3-i}), p_i^2(u_{3-i}, b_{3-i})/2)$$
(or if the latter inequality holds with equality, the mechanism can choose to allocate 2 units to i);
- *else 1 unit at price $p_i^1(u_{3-i}, b_{3-i})$ if*
$$b_i \ge p_i^1(u_{3-i}, b_{3-i})$$
and
$$u_i > p_i^1(u_{3-i}, b_{3-i})$$
(or if the latter inequality holds with equality, the mechanism can choose to allocate 1 units to i);
- *else 0 units.*

Conversely, for any set of threshold function $p_i^j : \Re_+^2 \to \Re_+ \cup \{\infty\}$, $1 \le i, j \le 2$, the mechanism defined above satisfies incentive compatibility and individual rationality.

PROOF. We prove the statement for $i = 1$ ($i = 2$ is analogous). Consider the threshold functions given by Proposition 3.1. Fix any bid (u_2, b_2) of the second agent. Suppose the true utility and budget of the first agent is u_1 and b_1, respectively. For simplicity, we use the notation $p_1^1 := p_1^1(u_2, b_2)$ and $p_1^2 := p_1^2(u_2, b_2)$. Notice that by the definition of p_1^1 and p_1^2 in the proof of Proposition 3.1, $p_1^1 \le p_1^2$. The first agent's utility for an allocation of 0 units is 0, 1 unit is $u_1 - p_1^1$ assuming $b_1 \ge p_1^1$, and 2 units is $2u_1 - p_1^2$ assuming $b_1 \ge p_1^2$. The first agent receives two units if and only if he has enough budget to pay for it (i.e., $b_1 \ge p_1^2$), and his utility for receiving two units ($2u_1 - p_1^2$) is greater than or equal to his utility for receiving one unit ($u_1 - p_1^1$) and zero units (zero). This can be written as $u_1 \ge p_1^2 - p_1^1$ and $u_1 \ge p_1^2/2$, or equivalently, $u_1 \ge p_1^2 - \min(p_1^1, p_1^2/2)$. Otherwise, if the first agent does not receive two units, then he receives one unit if and only if he has the budget (i.e., $b_1 \ge p_1^1$), and his utility for one unit ($u_1 - p_1^1$) is greater than or equal to his utility for zero units, or equivalently, $u_1 \ge p_1^1$. If these conditions do not hold, then the agent receives zero units. The converse follows easily from the definition of the mechanism. $\square$

4. AN IMPOSSIBILITY RESULT

In this section, we show that there is no deterministic truthful mechanism satisfying three properties which we define in this section, even if there are only two buyers and two units of the good.[2] This result automatically generalizes to auctions with more buyers, by considering the situation where all but two of the buyers bid zero.

The first property is the following. This is similar to a property with the same name defined by Moulin [13] in the context of group-strategyproof mechanisms for cost sharing problems.

- *consumer sovereignty* – For any agent i and any vector of bids (u_{-i}, b_{-i}) for other agents, there is a bid (u_i, b_i) such that if agents bid according to (u, b), then agent i receives all units of the good.

[2]Our result extends to randomized auctions that are strategyproof in the following stronger sense: no matter what the outcome of the coin flips are, it is a dominant strategy for the participants to reveal their true type. The randomized algorithm given in Section 5 is strategyproof in this sense only if the good is assumed to be divisible.

Intuitively, consumer sovereignty requires that each agent must be able to win all units if he bids *high enough*. This precludes trivial mechanisms that for example sell at most one unit to each bidder. In terms of the characterization in Proposition 3.1 and Corollary 3.1, this property is equivalent to saying that the threshold functions p_i^j are all finite.

The second property, which we call the *independence of irrelevant alternatives* (IIA), is a much weaker version of a property of the same name in Lavi et al. [11]. This property is defined as follows.

- *independence of irrelevant alternatives (IIA)* – For any agent i and a bid vector (u, b), if i receives no units at (u, b), then the allocation when every agent bids according to (u, b) is the same as the allocation when agent i bids $(0, 0)$ and others bid according to (u_{-i}, b_{-i}).

Intuitively, the above property states that if an agent who does not win the auction leaves, the allocation to other agents should not change (Their payment, however, might change). As we will see in the proof of Theorem 4.1, in the case of two buyers and two units, IIA is equivalent to the property that if bids of both agents are large enough (both the utility and the budget), then both units are allocated.

As we will see at the end of this section, there are truthful mechanisms not satisfying the IIA. In fact, the following example shows that even with IIA, there are mechanisms that are truthful.

EXAMPLE 4.1. *Bundling mechanism: Consider the mechanism that always* bundles *the two units, i.e., it allocates both units to the agent i such that* $\min(2u_i, b_i) > \min(2u_{3-i}, b_{3-i})$, *and charges him* $\min(2u_{3-i}, b_{3-i})$. *It is easy to see that this mechanism is truthful and satisfies the IIA.*

However, we conjecture that the bundling mechanism is essentially the only truthful mechanism satisfying the above properties. In other words, we would like to show that there is no truthful mechanism satisfying the above properties and the following.

- *non-bundling* – there is a bid vector (u, b) such that the mechanism allocates one unit of the good to each buyer.

Unfortunately, we do not know how to prove this conjecture. However, we can prove this statement under the following stronger condition.

- *strong non-bundling* – for any non-zero bid (u_1, b_1) of the first agent, there is a bid (u_2, b_2) for the second agent such that if both agents bid according to (u, b), the mechanism allocates one unit of the good to each buyer.

The following theorem is the main result of this section.

THEOREM 4.1. *There is no deterministic truthful auction for two buyers and two units of a good that satisfies consumer sovereignty, IIA, and strong non-bundling.*

PROOF. The proof is based on examining functional relations imposed by our assumptions on the threshold functions of any truthful auction. We obtain the impossibility result by showing that this set of functional relations has no solution.

The fact that our auction observes supply limits implies that whenever the threshold functions are such that the first (second) agent gets two units, then the second (first) agent must get zero units. The consumer sovereignty and IIA assumptions imply that these two situations are in fact equivalent in certain regions of the bid space, i.e., the mechanism always allocates all the units when the bids are large enough.

By consumer sovereignty, for each agent $i = 1, 2$, there is a bid (u_i^*, b_i^*) such that if i bids (u_i^*, b_i^*) and the other agent bids $(0, 0)$, then agent i wins both units. Furthermore, by Corollary 3.1, for every $u_i' \geq u_i^*$ and $b_i' \geq b_i^*$, if i bids (u_i', b_i') and the other agent bids $(0, 0)$, then i wins both units. Let $C = \max\{u_1^*, b_1^*, u_2^*, b_2^*\}$.

CLAIM 4.1. *For any set of bids* (u_1, b_1) *and* (u_2, b_2) *such that* $u_1, b_1, u_2, b_2 \geq C$, *the mechanism allocates both units when agents bid according to* (u, b). *Furthermore, the payment of any agent that receives at least one unit in this situation is non-zero.*

PROOF. Assume, for contradiction, that for one such bid vector the mechanism allocates at most one unit of the good to the first agent and zero units to the second agent. Now, by IIA, if the second agent bids $(0, 0)$, the first agent must still receive at most one unit. This, contradicts the definition of C. Now, assume that an agent, say 1, receives at least one unit but has to pay 0. This means that if agent 1 bids $(0, 0)$, he still wins at least one unit, and therefore agent 2 does not receive both units. This contradicts the definition of C. □

Immediate from Corollary 3.1 is the fact that the allocations and payments given bid (α_i, β_i) holding bid $(\alpha_{3-i}, \beta_{3-i})$ fixed is constant for all $\beta_i \geq 2\alpha_i$ and for all $\alpha_i \geq \beta_i$. We will use this observation to make statements about the properties of the threshold functions as one of the inputs becomes irrelevant (i.e., sufficiently large). Let

$$\begin{aligned} r_i^j(x) &= p_i^j(x, 2x), \\ s_i^j(x) &= p_i^j(x, x) \end{aligned}$$

for $i, j = 1, 2$. By Corollary 3.1, all of the above functions are non-decreasing functions. Therefore, they can be discontinuous in at most a countable number of points. Let T denote the set of points greater than C at which all of the above functions are continuous. Notice that since the number of discontinuity points of each of these functions is countable, the set T is dense in (C, ∞).

Claim 4.1 together with our characterization, Corollary 3.1, immediately imply the following functional relations:

LEMMA 4.1. *For all* $A, B \in T$,

$$B < r_{3-i}^1(A) \Rightarrow A \geq (s_i^2 - \min(s_i^1, s_i^2/2))(B) \tag{1}$$

PROOF. Suppose agent i bids $(A, 2A)$ and agent $(3-i)$ bids (B, B) and $B < r_{3-i}^1(A)$. Then agent $(3-i)$ receives zero units, so agent i must receive two units. As agent i's budget is essentially unconstrained, this implies that his utility is at least the utility threshold, or $A \geq (s_i^2 - \min(s_i^1, s_i^2/2))(B)$. □

Similarly, we can prove the following statements for every $A, B \in T$:

$$A > (s_i^2 - \min(s_i^1, s_i^2/2))(B) \Rightarrow B < r_{3-i}^1(A), \tag{2}$$

$$B \geq r^2_{3-i}(A) \Rightarrow A \leq \min(s^1_i, s^2_i/2)(B), \quad (3)$$

$$A < \min(s^1_i, s^2_i/2)(B) \Rightarrow B \geq r^2_{3-i}(A), \quad (4)$$

$$B > (r^2_{3-i} - \min(r^1_{3-i}, r^2_{3-1}/2))(A) \Rightarrow A \leq \min(r^1_i, r^2_i/2)(B), \quad (5)$$

$$A < \min(r^1_i, r^2_i/2)(B) \Rightarrow B \geq (r^2_{3-i} - \min(r^1_{3-i}, r^2_{3-1}/2))(A), \quad (6)$$

$$B \geq s^2_{3-i}(A) \iff A < s^1_i(B). \quad (7)$$

From these functional relations, we can derive the following inequalities.

LEMMA 4.2. *For all $A \in T$,*

$$(r^2_2 - \min(r^1_2, r^2_2/2))(A) \geq (s^2_2 - \min(s^1_2, s^2_2/2))(A). \quad (8)$$

PROOF. Choose $B \in T$, $B > (r^2_2 - \min(r^1_2, r^2_2/2))(A)$. Then relation 5 (with $i = 1$) implies $A \leq \min(r^1_1, r^2_1/2)(B) \leq r^1_1(B)$. Take $\epsilon > 0$ and note that relation 1 (with $i = 2$) implies $B > (s^2_2 - \min(s^1_2, s^2_2/2))(A - \epsilon)$. Taking the limit as ϵ goes to zero and using the continuity of s^1_2 and s^2_2 at A, we have that $B > (r^2_2 - \min(r^1_2, r^2_2/2))(A)$ implies $B \geq (s^2_2 - \min(s^1_2, s^2_2/2))(A)$. Since this statement holds for every $B \in T$ and T is dense, the lemma follows. $\square$

Similarly, we prove the following lemma.

LEMMA 4.3. *For all $A \in T$,*

$$\min(r^1_2, r^2_2/2)(A) \geq \min(s^1_2, s^2_2/2)(A). \quad (9)$$

PROOF. Choose $B \in T$, $B > \min(r^1_2, r^2_2/2)(A)$. By the contrapositive of relation 5, $A \leq (r^2_1 - \min(r^1_1, r^2_1/2))(B)$. By Claim 4.1, $\min(r^1_1, r^2_1/2)(B) > 0$. Hence, $A < r^2_1(B)$. This, by the contrapositive of relation 4, implies that $B \geq \min(s^1_2, s^2_2/2)(A)$. Since this holds for every $B \in T$ and T is dense, the lemma follows. $\square$

Our non-bundling assumption implies that for all $Z \in T$ the interval $(r^1_2(Z), r^2_2(Z))$ is non-empty. Select a point t in this interval and observe that the contrapositive of relations 2 (with $i = 1$) implies

$$Z \leq (s^2_1 - \min(s^1_1, s^2_1/2))(t). \quad (10)$$

Let $\epsilon > 0$ and note that t is in the interval $(r^1_2(Z-\epsilon), r^2_2(Z-\epsilon))$ for small enough ϵ by continuity. Thus the contrapositive of relation 4 with $i = 1$ implies

$$Z > Z - \epsilon \geq \min(s^1_1, s^2_1/2)(t). \quad (11)$$

Combining inequalities 10 and 11, we get

$$(s^2_1 - \min(s^1_1, s^2_1/2))(t) > \min(s^1_1, s^2_1/2)(t)$$

and so

$$\min(s^1_1, s^2_1/2)(t) = s^1_1(t). \quad (12)$$

Equations 11 and 12 imply that for every $t \in (r^1_2(Z), r^2_2(Z))$, $Z > s^1_1(t)$. By Equation 7, this implies that $t < s^2_2(Z)$. Taking the limit of this equation as t tends to $r^2_2(Z)$, we obtain $r^2_2(Z) \leq s^2_2(Z)$. On the other hand, summing Equations 8 and 9 implies that $r^2_2(Z) \geq s^2_2(Z)$. Therefore, $r^2_2(Z) = s^2_2(Z)$. Thus, inequalities 8 and 9 must both attain equality at Z. Ranging over choice of $Z \in T$, we see that inequalities 8 and 9 must attain equality everywhere in T. Our contradiction arises from the observation that in fact for some $Z \in T$, inequality 9 is strict. By Claim 4.1, prices are always nonzero, and so $C < (r^2_1 - \min(r^1_1, r^2_1/2))(A) < r^2_1(A)$ for some $A \in T$. Select such an A and $Z \in ((r^2_1 - \min(r^1_1, r^2_1/2))(A), r^2_1(A)) \cap T$. Notice that since T is a dense set, this intersection is nonempty. Note that relation 4 with $i = 2$ implies that $A \geq \min(s^1_2, s^2_2/2)(Z)$. Therefore, for any small $\epsilon > 0$, $A + \epsilon > \min(s^1_2, s^2_2/2)(Z)$. Similarly, note that relation 5 with $i = 2$ implies $A + \epsilon \leq \min(r^1_2, r^2_2/2)(Z)$ for ϵ sufficiently small. But this means that, for this particular Z, $\min(r^1_2, r^2_2/2)(Z) > \min(s^1_2, s^2_2/2)(Z)$, yielding our contradiction. $\square$

The following example shows that the IIA assumption in Theorem 4.1 is necessary, i.e., there are deterministic truthful mechanisms that satisfy consumer sovereignty and non-bundling, but not IIA.

EXAMPLE 4.2. *Consider an auction with 2 units and 2 bidders which uses the following rules for allocation to agent i ($i = 1, 2$):*

- *If $u_i > 2\min(u_{3-i}, b_{3-i})$ and $b_i > \frac{5}{2}\min(u_{3-i}, b_{3-i})$, then agent i gets 2 units and pays $\frac{5}{2}\min(u_{3-i}, b_{3-i})$;*
- *else if $u_i > \frac{1}{2}\min(u_{3-i}, b_{3-i})$ and $b_i > \frac{1}{2}\min(u_{3-i}, b_{3-i})$, then agent i gets 1 unit and pays $\frac{1}{2}\min(u_{3-i}, b_{3-i})$;*
- *else agent i receives nothing.*

It is not hard to verify that this mechanism satisfies the characterization given in Section 3, and is therefore truthful. However, if, for example, agent 1 bids $(4a, 4a)$ and agent 2 bids $(9a, 9a)$ for any a, the mechanism allocates zero units to agent 1 and one unit to agent 2. Therefore, the mechanism does not allocate both units even if bids are sufficiently large, and hence it does not satisfy IIA.

5. AN ASYMPTOTICALLY OPTIMAL AUCTION

As we saw in the previous sections, there appears to be no reasonable mechanism for allocating all units truthfully. In this section, we consider mechanisms that may allocate only some of the units, and among them, seek the one that maximizes the expected revenue. We do this by designing a mechanism that has revenue comparable to the maximum that can be achieved by a posted-price auction.

The method that we use for design and analysis of our auction is inspired by the work of Goldberg et al. [7, 8]. As in [7, 8], we take the *competitive ratio* to be the expected revenue of our mechanism over the revenue of the optimum posted-price auction, and attempt to design an auction which minimizes this ratio. However, our design analysis differs from that of [7, 8] in some important aspects. Unlike [7, 8] and subsequent results, in our setting the mechanism may need to allocate more than one unit to every agent. Moreover, the agents can lie about both their bid and their budget, which introduces significant complications.

5.1 A Truthful Mechanism

Our mechanism is quite natural: similar to the random sampling optimal threshold auction of [7, 8], we divide bidders into two random subsets, compute the optimal price for each subset, and offer that price to the other subset. In order to guarantee that our auction doesn't oversell the good, we sell at most half the available units to each subset, greedily allocating units to interested agents arranged in an arbitrary order.

Note, although our units are indivisible, we can assume that fractional allocations are possible by using the proper randomization: whenever the algorithm asks us to allocate a fraction c of a unit to an agent, we instead charge the agent c times the offering price for participation in a lottery that offers him a full unit with probability c. Thus an agent's payment is deterministic and his expected utility is constant. Only his allocation is randomized. For the remainder of this section, we assume without loss of generality that our units are divisible.

As before, let n be the number of agents and m be the number of available units of a good. Each agent i submits his utility value for one unit u_i and his maximum budget b_i.

Mechanism

- Partition the agents randomly into two sets A and B by independently putting each agent into either set uniformly at random with probability $\frac{1}{2}$.
- From the set of utility values u_i of agents $i \in A$, choose p_A to be the price which maximizes the revenue of selling at most $m/2$ units in A. In other words, if the u_i's are sorted in decreasing order, for
$$i_0 = \min\left\{ i : \sum_{j=1}^{i-1} b_j \geq \frac{u_i m}{2} \right\},$$
define $p_A = u_{i_0-1}$. Compute p_B analogously.
- Consider the agents in A in a random order and allocate at most $\frac{m}{2}$ units to them as follows. In every step, if the utility of agent i satisfies $u_i \geq p_B$, allocate $\frac{b_i}{p_B}$ units to i, or all remaining units if less than $\frac{b_i}{p_B}$ units remain. Charge i a price of p_B per unit. Apply the same procedure to the set B using the threshold value p_A.

First, we give a simple proof of the truthfulness of the mechanism.

LEMMA 5.1. *The above mechanism is truthful, i.e., for every agent reporting the correct utility and budget values is a dominant strategy.*

PROOF. Consider an agent i in A. First we argue that agent i does not have any incentive to misreport his utility value. We know that agent i receives a unit only if $u_i \geq p_B$, and that he pays p_B for a unit if he receives it. The two key observations are that (1) the threshold p_B is determined independently of all u_j and b_j, $j \in A$, including $j = i$; and (2) when the supply of units in A is inadequate to meet the demands of all agents in A whose utilities exceed p_B, then the allocation of units to those agents is done in an arbitrary order, again independently of all u_j and b_j.

Finally, by reporting a budget below b_i, agent i would potentially decrease his allocation and hence his total utility. By reporting a budget above b_i, if he was previously saturating his budget, then his allocation might increase causing him to be charged more than his budget and decreasing his total utility to negative infinity. Otherwise, if he was not previously saturating his budget, his allocation will not change. In either case, he has no incentive to misreport. □

5.2 Revenue guarantee

The truthfulness of this mechanism is straightforward, but providing a revenue guarantee requires a more careful analysis. An important parameter for the analysis is what we call the *budget dominance parameter*, or the ratio of the maximum budget of an agent to the revenue of the optimum posted-price auction. It is not surprising that the revenue guarantee is a function of the budget dominance parameter; indeed, it is imposed upon us by the condition of truthfulness. In a setting with a large budget-dominance parameter, the revenue of the optimum posted-price is dominated by a single agent. In order to extract the full budget of this agent, the price offered to him must be less than his reported utility and thus is affected by his bid.

The mechanism we design has the property that its revenue approaches that of the optimum posted-price auction as the budget dominance parameter tends to 0. In particular, we will prove that for all $0 < \delta < 1$, the revenue of our mechanism is at least a $(1-\delta)$ fraction of the optimum posted-price revenue with probability at least $1 - O(e^{-c\delta^2/\epsilon})$ where c is some constant and ϵ is the budget-dominance parameter. Our proof is fairly natural. We first show that, using a price which is at least the optimum posted-price and disregarding supply limits, the revenue extracted from each random subset of bidders is approximately equal. We will use this to claim that the revenue extracted by our mechanism from each subset is almost half the optimum.

For notational convenience, without loss of generality, we assume that $u_1 > u_2 > \cdots > u_n$. For any price p, we denote by $r_S(p, k)$ the revenue of allocating at most k units to a set S of agents at price p:
$$r_S(p, k) = \min(kp, \sum_{\substack{j \in S \\ u_j \geq p}} b_j).$$
We will use the notation $r(p, k)$ in the case where we are allocating the units to the whole set $A \cup B$. Finally, we also define $r(p) = r(p, \infty) = \sum_{u_i \geq p} b_i$.

Given the utility and budget values of the agents, one can find the optimum price $p*$ at which $r(p, m)$ is maximized, and allocate the units at this price. We call this mechanism the optimum posted-price auction $OPT = r(p^*, m)$. In our argument, we will use the following properties of an optimum posted-price auction for allocating at most k units, for any k.

1. There exists an agent i such that selling the units at price $p = u_i$ results in the optimum revenue.
2. For any k, if p is the optimum price for allocating at most k units, then $r(p, k) \leq r(p) \leq r(p, k) + b_{max}$ where $b_{max} = \max_i b_i$. In particular,
$$OPT \leq r(p^*) \leq OPT + b_{max}.$$

Let ϵ denote the *budget dominance parameter*, that is the ratio of the maximum budget of all agents, b_{max}, to the value of the optimum solution OPT. As we will show, the probability of success of our algorithm is asymptotically controlled by ϵ. The next lemma shows that the revenue extracted from each subset at or above the optimum price is approximately equal, disregarding supply limits, with probability approaching 1 as ϵ approaches 0.

LEMMA 5.2. *Let $\delta > 0$. Then the probability that*

$$\Big|r_A(u_l) - r_B(u_l)\Big| < \delta OPT \quad \text{for all} \quad l \quad \text{with} \quad u_l \geq p^*$$

is at least $1 - 2e^{-\delta^2/(4\epsilon)}$.

PROOF. Define α_i to be a random variable indicating whether agent i is in A, with $\alpha_i = 1$ when $i \in A$ and $\alpha_i = -1$ when $i \in B$. Let $S_i = \sum_{j \leq i} \alpha_j b_j$. Then $|r_A(u_l) - r_B(u_l)| = |S_l|$. Thus we need to bound the probability that the random variable S_i deviates by more than δOPT from its expectation 0.

Let $\tau(\delta) = \min\{i : |S_i| \geq \delta OPT\}$. We define the following martingale:

$$\tilde{S}_i = \begin{cases} S_i & \text{if } i \leq \tau(\delta) \\ S_{\tau(\delta)} & \text{otherwise} \end{cases}$$

Let k be such that $u_k = p^*$. Then we have

$$\begin{aligned} 1 - \Pr\left(|r_A(u_i) - r_B(u_i)| < \delta OPT,\ \forall i \leq k\right) & \\ = \quad & 1 - \Pr\left(|S_i| < \delta OPT,\ \forall i \leq k\right) \\ = \quad & \Pr\left(\exists\ i \leq k\ :\ |S_i| \geq \delta OPT\right) \\ = \quad & \Pr\left(\tau(\delta) \leq k\right) \\ = \quad & \Pr\left(|\tilde{S}_k| \geq \delta OPT\right) \end{aligned}$$

Now since $\tilde{S}_i$ is a martingale, by the Azuma-Hoeffding inequality we have:

$$\Pr(|\tilde{S}_k| \geq \delta OPT) \leq 2\exp\left(\frac{-\delta^2 OPT^2}{2\sum_{i \leq k} b_i^2}\right).$$

Bounding the sum $\sum_{i \leq k} b_i^2$ by $b_{max} r(p^*) \leq \epsilon OPT r(p^*)$ and using that $r(p^*) \leq OPT(1+\epsilon) \leq 2OPT$, we obtain the lemma. □

From now on, we will say that an event happens with high probability if its probability is at least $1 - 2e^{-\delta^2/(4\epsilon)}$. From the previous lemma, it is clear that the revenue of each subset at the optimum price is almost half the optimum revenue with high probability, disregarding supply limits. In fact, it is not hard to see that this statement holds even *observing supply limits*.

COROLLARY 5.1. *With high probability, we have*

$$r_A(p^*, \frac{m}{2}) \geq \frac{1-\delta}{2} OPT.$$

PROOF. First note $OPT = r(p^*, m) = \min\{p^* m, r(p^*)\}$, implying $mp^* \geq OPT$ and $r(p^*) \geq OPT$. From Lemma 5.2, we have with high probability, $r_A(p^*) \geq r(p^*) - \frac{\delta}{2} OPT \geq \frac{1-\delta}{2} OPT$. Furthermore, $\frac{m}{2} p^* \geq \frac{1-\delta}{2} OPT$ by definition of OPT, so $r_A(p^*, \frac{m}{2}) = \min(\frac{m}{2} p^*, r_A(p^*)) \geq \frac{1-\delta}{2} OPT$. □

At this point we would be done if our mechanism computed the offering price p^*. Unfortunately, we can not compute this price. Instead we compute prices p_A and p_B, the optimal prices for subsets A and B, and offer these prices to the opposing set. Thus we need to a prove statement similar to that of Lemma 5.2 for all offering prices. The following corollary states that for *any* offering price, the revenue of one subset is either close to the revenue of the other subset or close to half the optimum revenue, disregarding supply limits.

COROLLARY 5.2. *With high probability, we have that*

$$r_B(u_k) \geq \min\{r_A(u_k) - \delta OPT, \frac{1-\delta}{2} OPT\} \qquad \text{for all} \quad k.$$

PROOF. By Lemma 5.2, we have that with high probability,

$$r_B(u_k) \geq r_A(u_k) - \delta OPT \quad \text{for all } k \text{ with } u_k \geq p^*. \tag{13}$$

Recalling that $r_A(p^*) + r_B(p^*) = r(p^*) \geq OPT$, we conclude that with high probability, both (13) and

$$r_B(p^*) \geq \frac{1-\delta}{2} OPT$$

hold simultaneously. By monotonicity, this implies the statement of the lemma. Indeed, either $u_k \geq p^*$ so that $r_B(u_k) \geq r_A(u_k) - \delta OPT$, or $u_k \leq p^*$ and $r_B(u_k) \geq r_B(p^*) \geq \frac{1-\delta}{2} OPT$, which gives the lemma. □

Finally, we have developed all the necessary machinery to prove the main theorem.

THEOREM 5.1. *The mechanism described in the previous section is truthful. Furthermore, for all $0 < \delta < 1$, the algorithm has revenue at least $(1-\delta)OPT$ with probability $1 - O(e^{-c\delta^2/\epsilon})$ for some constant c and $\epsilon = b_{max}/OPT$.*

PROOF. Recall that p_A is the price which maximizes the revenue of selling at most $\frac{m}{2}$ units in A. Thus, for all p, we have $r_A(p, \frac{m}{2}) \leq r_A(p_A, \frac{m}{2})$, so in particular $r_A(p_A, \frac{m}{2}) \geq r_A(p^*, \frac{m}{2})$. Combined with Corollary 5.1, we conclude that with high probability, $r_A(p_A, \frac{m}{2}) \geq \frac{1-\delta}{2} OPT$, which in turn implies that

$$p_A \frac{m}{2} \geq \frac{1-\delta}{2} OPT \tag{14}$$

and

$$r_A(p_A) \geq \frac{1-\delta}{2} OPT. \tag{15}$$

Combined with Corollary 5.2, inequality 15 gives

$$r_B(p_A) \geq \frac{1-3\delta}{2} OPT, \tag{16}$$

again with high probability. Inequalities 14 and 16 together with the definition of $r_B(p_A, \frac{m}{2})$ imply that with high probability,

$$r_B(p_A, \frac{m}{2}) \geq \frac{1-3\delta}{2} OPT.$$

Exchanging the roles of A and B, we get the same result for $r_A(p_B, \frac{m}{2})$. Since $r_B(p_A, \frac{m}{2}) + r_A(p_B, \frac{m}{2})$ is the revenue of the algorithm, this establishes the theorem. □

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