Integer Programming and Arrovian Social Welfare Functions

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Abstract

We formulate the problem of deciding which preference domains admit a non-dictatorial Arrovian Social Welfare Function as one of verifying the feasibility of an integer linear program. Many of the known results about the presence or absence of Arrovian Social Welfare Functions, impossibility theorems in Social Choice theory, and properties of majority rules etc., can be derived in a simple and unified way from this integer program. We characterize those preference domains that admit a non-dictatorial, neutral Arrovian Soical Welfare function and give a polyhedral characterization of Arrovian Social Welfare Functions on single-peaked domains.

Keywords: Social Welfare Function, Impossibility Theorem, Single-Peaked Domain, Linear Programming

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1 Introduction

The Old Testament likens the generations of men to the leaves of a tree. It is a simile that applies as aptly to the literature inspired by Arrow's impossibility theorem [2]. Much of it is devoted to classifying those preference domains that admit or exclude the existence of a non-dictatorial Arrovian Social Welfare Function (ASWF).[1] We add another leaf to that tree. Here we formulate the problem of deciding whether a preference domain admits a non-dictatorial ASWF as an integer program. This formulation allows us to derive in a systematic way many of the known results about Arrovian domains. It is inspired by a characterization of Arrovian domains due to Kalai and Muller [8].

Let $\mathcal{A}$ denote the set of alternatives (at least three). Let Σ denote the set of all transitive, antisymmetric and total binary relations on $\mathcal{A}$. An element of Σ is a preference ordering. Notice that this set up excludes indifference. The set of admissible preference orderings for members of a society of n-agents (voters) will be a subset of Σ and denoted Ω. Let Ω^n be the set of all n-tuples of preferences from Ω, called **profiles**. An element of Ω^n will typically be denoted as $\mathbf{P} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n})$, where $\mathbf{p_i}$ is interpreted as the preference ordering of agent i. In the language of Le Breton and Weymark [10], we assume the **common preference domain** framework.

An n-person Social Welfare Function is a function $f : \Omega^n \to \Sigma$. Thus for any $\mathbf{P} \in \Omega^n$, $f(\mathbf{P})$ is an ordering of the alternatives. We write $xf(\mathbf{P})y$ if x is ranked above y under $f(\mathbf{P})$. An n-person **Arrovian Social Welfare Function** (ASWF) on Ω is a function $f : \Omega^n \to \Sigma$ that satisfies the following two conditions:

1. **Unanimity:** If for $\mathbf{P} \in \Omega^n$ and some $x, y \in \mathcal{A}$ we have $x\mathbf{p_i}y$ for all i then $xf(\mathbf{P})y$.
2. **Independence of Irrelevant Alternatives:** For any $x, y \in \mathcal{A}$ suppose $\exists \mathbf{P}, \mathbf{Q} \in \Omega^n$ such that $x\mathbf{p_i}y$ if an only if $x\mathbf{q_i}y$ for $i = 1, \ldots, n$. Then $xf(\mathbf{P})y$ if an only if $xf(\mathbf{Q})y$.

The first axiom stipulates that if all voters prefer alternative x to alternative y, then the social welfare function f must rank x above y. The second axiom states that the ranking of x and y in f is not affected by how the voters rank the other alternatives. An obvious Social Welfare function that satisfies the two conditions is the *dictatorial rule*: rank the alternatives in the order of the preferences of a particular voter (the dictator). Formally, an ASWF is **dictatorial** if there is an i such that $f(\mathbf{P}) = \mathbf{p_i}$ for all $\mathbf{P} \in \Omega^n$. An ordered pair $x, y \in \mathcal{A}$ is called **trivial** if $x\mathbf{p}y$ for all $\mathbf{p} \in \Omega$. In view of unanimity, any ASWF must have $xf(\mathbf{P})y$ for all $\mathbf{P} \in \Omega^n$ whenever x, y is a trivial pair. If Ω consists only of trivial pairs then distinguishing between dictatorial and non-dictatorial ASWF's becomes nonsensical, so we assume that Ω contains at least one non-trivial pair. The domain Ω is **Arrovian** if it admits a non-dictatorial ASWF.

The conditions identified by Kalai and Muller for the existence of a 2-person non-dictatorial ASWF have a natural interpretation as an integer programming problem. In fact, all 2-person

[1] An ASWF is a social welfare function that satisfies the axioms of the Impossibility theorem.

ASWF's are solutions to this integer program. A natural question that arises is whether there exists an efficient characterization of all n-person ASWF's. In this paper, we address this question by observing that the axioms for ASWF's induce a natural integer programming formulation. This approach, while intuitive, allows us to derive several structural and impossibility theorems in social choice theory in a unified and simple way.

The main contributions of this paper are summarized below.

- Our first result is an integer linear programming formulation of the problem of finding a n-person ASWF. For each Ω we construct a set of linear inequalities with the property that every feasible 0-1 solution corresponds to a n-person ASWF. The formulation is an extension of the conditions identified by Kalai and Muller's [8] for the case $n = 2$ to general n. In fact, the characterization extends easily to the case when the common preference domain assumption is dropped.

- When restricted to the class of neutral ASWF's the integer program yields a simple and easily checkable characterization of domains that admit neutral, non-dictatorial ASWF's. This result contains as a special case the results of Sen [17] and Maskin [11] about the robustness of majority rule.

- For the case when Ω is single-peaked, we show that the polytope defined by the set of linear inequalities is **integral**: the vertices of the polytope correspond to ASWF's and every ASWF corresponds to a vertex of the polytope. This gives the first characterization of ASWF's on this domain we are aware of. The same proof technique yields a characterization of the generalized majority rule on single peaked domains, originally due to Moulin [13].

- To illustrate the versatility of the integer program, we use it to derive dictatorship results for social choice functions under monotonicity (Muller and Satterthwiate [14]) and strategic candidacy (Dutta, Jackson and Le Breton [5]). We argue, by the weight of examples, that the integer programming approach allows one to derive in a unified, systematic and transparent way a whole range of results in social choice theory.

- In a different vein, we point out that the computational complexity of deciding whether a domain is Arrovian depends critically on the way the domain is described. In fact, the integer programming formulation implied by Kalai and Muller [8] cannot even be explicitly determined in polynomial time (unless $P = NP$), let alone checking feasibility of non-trivial integral solution. We also propose a graph-theoretical method to identify stronger linear inequalities for ASWF's. For cases with a small number of alternatives (3 or 4), our approach is able to characterize the polytope of all ASWF's. Thus for *any* Ω and any set of alternatives size at most 4 we can characterize the polyhedral structure of all ASWF's.

2 The Integer Program

Denote the set of all ordered pairs of alternatives by $\mathcal{A}^2$. Let E denote the set of all agents, and S^c denote $E \setminus S$ for all $S \subseteq E$.

To construct an n-person ASWF we exploit the independence of irrelevant alternatives condition. This allows us to specify an ASWF in terms of which ordered pair of alternatives a particular subset, S, of agents is decisive over.

Definition 1 *For a given ASWF f, a subset S of agents is* **weakly decisive for** x **over** y *if whenever all agents in S rank x over y and all agents in S^c rank y over x, the ASWF f ranks x over y.*

Since this is the only notion of decisiveness used in the paper, we omit the qualifier 'weak' in what follows.

For each non-trivial element $(x, y) \in \mathcal{A}^2$, we define a 0-1 variable as follows:

$$d_S(x, y) = \begin{cases} 1, & \text{if the subset } S \text{ of agents is decisive for } x \text{ over } y; \\ 0, & \text{otherwise.} \end{cases}$$

If $(x, y) \in \mathcal{A}^2$ is a trivial pair then by default we set $d_S(x, y) = 1$ for all $S \neq \emptyset$.

Given an ASWF f, we can determine the associated d variables as follows: for each $S \subseteq E$, and each non-trivial pair (x, y), pick a $\mathbf{P} \in \Omega^n$ in which agents in S rank x over y, and agents in S^c rank y over x; if $xf(\mathbf{P})y$, set $d_S(x, y) = 1$, else set $d_S(x, y) = 0$.

In the rest of this section, we identify some conditions satisfied by the d variables associated with an ASWF f.

<u>**Unanimity**</u>: To ensure unanimity, for all $(x, y) \in \mathcal{A}^2$, we must have

$$d_E(x, y) = 1. \tag{1}$$

<u>**Independence of Irrelevant Alternatives**</u>: Consider a pair of alternatives $(x, y) \in \mathcal{A}^2$, a $\mathbf{P} \in \Omega^n$, and let S be the set of agents that prefer x to y in $\mathbf{P}$. (Thus, each agent in S^c prefers y to x in $\mathbf{P}$.) Suppose $xf(\mathbf{P})y$. Let $\mathbf{Q}$ be any other profile such that all agents in S rank x over y and all agents in S^c rank y over x. By the independence of irrelevant alternatives condition $xf(\mathbf{Q})y$. Hence the set S is decisive for x over y. However, had $yf(\mathbf{P})x$ a similar argument would imply that S^c is decisive for y over x. Thus, for all S and $(x, y) \in \mathcal{A}^2$, we must have

$$d_S(x, y) + d_{S^c}(y, x) = 1. \tag{2}$$

A consequence of Eqs. (1) and (2) is that $d_\emptyset(x, y) = 0$ for all $(x, y) \in \mathcal{A}^2$.

<u>**Transitivity**</u>: To motivate the next class of constraints, it is useful to consider majority rule. If the number n of agents is odd, majority rule can be described using the following variables:

$$d_S(x, y) = \begin{cases} 1, & \text{if } \ |S| > n/2, \\ 0, & \text{otherwise.} \end{cases}$$

These variables satisfy both (1) and (2). However, if Ω admits a *Condorcet* triple (e.g., $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \in \Omega$ with $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$), then such a rule does not always produce an *ordering* of the alternatives for each preference profile. Our next constraint (**cycle elimination**) is designed to exclude this and similar possibilities.

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) \leq 2, \tag{3}$$

where the sets satisfy the following conditions (hereafter referred to as conditions (*)):

$$A \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}z\mathbf{p}y,$$
$$B \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}x\mathbf{p}z,$$
$$C \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}y\mathbf{p}x,$$
$$U \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}y\mathbf{p}z,$$
$$V \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}x\mathbf{p}y,$$
$$W \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}z\mathbf{p}x.$$

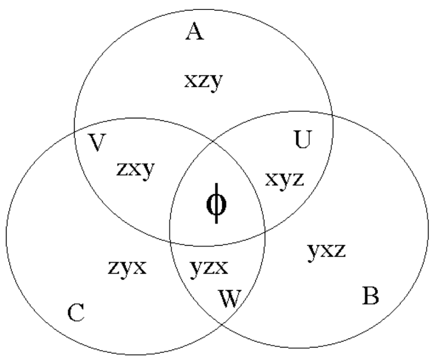

Figure 1: The sets and the associated orderings

The constraint ensures that on any profile $\mathbf{P} \in \Omega^n$, the ASWF f does not produce a ranking that "cycles".

A consequence of (2) and (3) that will be useful is that

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) \geq 1. \tag{4}$$

To deduce it, interchange the roles of z and x in (3). Then the roles of A and V (resp. B and W, C and U) can be interchanged to obtain the new inequality:

$$d_{A\cup C\cup V}(z,y) + d_{B\cup C\cup W}(y,x) + d_{A\cup B\cup U}(x,z) \leq 2.$$

Using (2), we obtain

$$d_{B\cup U\cup W}(y,z)+d_{A\cup U\cup V}(x,y)+d_{C\cup V\cup W}(z,x)\geq 1.$$

Subsequently we prove that constraints (1-3) are both necessary and sufficient for the characterization of n-person ASWF's. Before that, it is useful to develop a better understanding of constraints (3), and their relationship to the constraints identified in [8], called *decisiveness implications*, described below.

Suppose there are $\mathbf{p},\mathbf{q}\in\Omega$ and three alternatives x,y and z such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Kalai and Muller [8] showed that

$$d_S(x,y)=1\Rightarrow d_S(x,z)=1,$$

and

$$d_S(z,x)=1\Rightarrow d_S(y,x)=1.$$

These conditions can be formulated as the following two inequalities:

$$d_S(x,y)\leq d_S(x,z), \tag{5}$$

$$d_S(z,x)\leq d_S(y,x). \tag{6}$$

The first condition follows from using a profile $\mathbf{P}$ in which agents in S rank x over y over z and agents in S^c rank y over z over x. If S is decisive for x over y, then $xf(\mathbf{P})y$. By unanimity, $yf(\mathbf{P})z$. By transitivity, $xf(\mathbf{P})z$. Hence S is also decisive for x over z. The second condition follows from a similar argument.

Claim 1 *Constraints (5, 6) are special cases of constraints (3).*

Proof. Let

$$U\leftarrow S, W\leftarrow S^c$$

in constraint (3), with the other sets being empty. U and W can be assumed non-empty by condition (*). Constraint (3) reduces to

$$d_U(x,y)+d_{U\cup W}(y,z)+d_W(z,x)\leq 2.$$

Since $U\cup W=E$, the above reduces to

$$0\leq d_S(x,y)+d_{S^c}(z,x)\leq 1,$$

which implies $d_S(x,y)\leq d_S(x,z)$ by (2). By interchanging the roles of S and S^c, we obtain the inequality $d_{S^c}(x,y)\leq d_{S^c}(x,z)$, which is equivalent to $d_S(z,x)\leq d_S(y,x)$. ■

Suppose we know only that there is a $\mathbf{p}\in\Omega$ with $x\mathbf{p}y\mathbf{p}z$. In this instance, transitivity requires:

$$d_S(x,y)=1 \textbf{ and } d_S(y,z)=1\Rightarrow d_S(x,z)=1,$$

and

$$d_S(z,x) = 1 \Rightarrow \textbf{ at least one of } \ d_S(y,x) = 1 \textbf{ or } \ d_S(z,y) = 1.$$

These can be formulated as the following two inequalities:

$$d_S(x,y) + d_S(y,z) \leq 1 + d_S(x,z), \tag{7}$$

$$d_S(z,y) + d_S(y,x) \geq d_S(z,x). \tag{8}$$

Similarly, we have:

Claim 2 *Constraints (7, 8) are special cases of constraints (3).*

Proof. Suppose $\exists\ \mathbf{q} \in \Omega$ with $x\mathbf{q}y\mathbf{q}z$. If there exists $\mathbf{p} \in \Omega$ with $y\mathbf{p}z\mathbf{p}x$ or $z\mathbf{p}x\mathbf{p}y$, then constraints (7, 8) are implied by constraints (5, 6), which in turn are special cases of constraint (3). So we may assume that there does not exist $\mathbf{p} \in \Omega$ with $y\mathbf{p}z\mathbf{p}x$ or $z\mathbf{p}x\mathbf{p}y$. If there does not exist $\mathbf{p} \in \Omega$ with $z\mathbf{p}y\mathbf{p}x$, then x, z is a trivial pair, and constraints (7, 8) are redundant. So we may assume that such a $\mathbf{p}$ exists, hence C can be chosen to be non-empty. Let

$$U \leftarrow S, C \leftarrow S^c$$

in constraint (3), with the other sets being empty. Constraint (3) reduces to

$$d_U(x,y) + d_U(y,z) + d_C(z,x) \leq 2,$$

which is just

$$d_S(x,y) + d_S(y,z) + d_{S^c}(z,x) \leq 2.$$

Thus constraint (7) follows as a special case of constraint (3). By reversing the roles of S and S^c again, we can show that constraint (8) follows as a special case of constraint (3). ■

For $n = 2$, we can show that constraints (1-3) reduce to constraints (1, 2, 5-8). Thus, Constraints (3) generalize the decisiveness implication conditions to $n \geq 3$. We will sometimes refer to (1)-(3) as IP.

Theorem 1 *Every feasible integer solution to (1)-(3) corresponds to an ASWF and vice-versa.*

Proof. Given an ASWF, it is easy to see that the corresponding d vector satisfies (1)-(3). Now pick any feasible solution to (1)-(3) and call it d. To prove that d gives rise to an ASWF, we show that for every profile of preferences from Ω, d generates an ordering of the alternatives. Unanimity and Independence of Irrelevant Alternatives follow automatically from the way the d_S variables are used to construct the ordering.

Suppose d does not produce an ordering of the alternatives. Then, for some profile $\mathbf{P} \in \Omega^n$, there are three alternatives x, y and z such that d ranks x over y, y over z and z over x. For this to happen there must be three non-empty sets H, I, and J such that

$$d_H(x,y) = 1, \ d_I(y,z) = 1, \ d_J(z,x) = 1,$$

and for the profile $\mathbf{P}$, agent i ranks x over y (resp. y over z, z over x) if and only if i is in H (resp. I, J). Note that $H \cup I \cup J$ is the set of all agents, and $H \cap I \cap J = \emptyset$.

Let

$$A \leftarrow H \setminus (I \cup J), B \leftarrow I \setminus (H \cup J), C \leftarrow J \setminus (H \cup I),$$

$$U \leftarrow H \cap I, V \leftarrow H \cap J, W \leftarrow I \cap J.$$

Now A (resp. B, C, U, V, W) can only be non-empty if there exists $\mathbf{p}$ in Ω with $x\mathbf{p}z\mathbf{p}y$ (resp. $y\mathbf{p}x\mathbf{p}z$, $z\mathbf{p}y\mathbf{p}x$, $x\mathbf{p}y\mathbf{p}z$, $z\mathbf{p}x\mathbf{p}y$, $y\mathbf{p}z\mathbf{p}x$).

In this case constraint (3) is violated since

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) = d_H(x,y) + d_I(y,z) + d_J(z,x) = 3.$$

■

Suppose $\Omega \subset \Omega'$. Then the constraints of IP corresponding to Ω are a subset of the constraints of IP corresponding to Ω'. However, we cannot infer that that any ASWF for Ω' will specify an ASWF for Ω. For example, (x,y) may be a trivial pair in Ω but not in Ω'. In this case $d_S(x,y) = 1$ is an additional constraint in the integer program for Ω but not in the integer program for Ω'.

We now show how the integer programming formulation can be used to derive Arrow's Impossibility Theorem.

Theorem 2 (Arrow's Impossibility theorem) *When $\Omega = \Sigma$, the 0-1 solutions to the IP correspond to dictatorial rules.*

Proof: When $\Omega = \Sigma$, we know from constraints (5-6) and the existence of all possible triples that $d_S(x,y) = d_S(y,z) = d_S(z,u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x,y)$ in the rest of the proof.

We show first that $d_S = 1 \Rightarrow d_T = 1$ for all $S \subset T$. Suppose not. Let T be the set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$ and $V = S$ in (3). Then, $d_{A\cup U\cup V} = d_E = 1$, $d_{B\cup U\cup W} = d_{T^c} = 1$ and $d_{C\cup V\cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \emptyset$, otherwise the assignment $A = (S \cup T)^c, U = S, V = T$ will violate the cycle elimination constraint. Furthermore, $d_{S\cap T} = 1$, otherwise the assignment $A = (S \cup T)^c, U = T \setminus S, V = S \setminus T, W = S \cap T$ will violate the cycle elimination constraint. Hence there exists a minimal set S^* with $d_{S^*} = 1$ such that all T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not there will be $j \in S$ with $d_j = 0$, which by (2) implies $d_{E\setminus\{j\}} = 1$. Since $d_{S^*} = 1$ and $d_{E\setminus\{j\}} = 1$, $d_{E\setminus\{j\}\cap S^*} \equiv d_{S^*\setminus\{j\}} = 1$, contradicting the minimality of S^*. ■

For subsequent applications we introduce the **born loser** rule. For each j, we define the **born loser** rule with respect to j (denoted by B_j) in the following way:

- $d_E^{B_j}(x,y) = 1$ for every $x, y \in \mathcal{A}^2$.
- $d_\emptyset^{B_j}(x,y) = 0$ for every $x, y \in \mathcal{A}^2$.
- For every non-trivial pair (x, y), and for any $S \neq \emptyset, E$, $d_S^{B_j}(x,y) = 0$ if $S \ni j$, $d_S^{B_j}(x,y) = 1$ otherwise.

Theorem 3 *For any j and $n > 2$, the born loser rule B_j is a non-dictatorial n-person ASWF if and only if for all x, y, z, there do not exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with*

$$x\mathbf{p_1}z\mathbf{p_1}y, \ x\mathbf{p_2}y\mathbf{p_2}z, \ z\mathbf{p_3}x\mathbf{p_3}y,$$

Proof. It is clear that by definition, d^{B_j} satisfies (1, 2). To see that it satisfies (3), observe that in every partition of the agents, one of the sets obtained must contain j. Say $j \in A \cup U \cup V$. If $d_{A\cup U\cup V}^{B_j}(x,y) = 0$, then (3) is clearly valid. So we may assume that $d_{A\cup U\cup V}^{B_j}(x,y) = 1$. This happens only when $A \cup U \cup V = E$ (or if (x, y) is trivial, which in turns imply that all the other sets are empty). We may assume $U, V \neq \emptyset$ and $j \in A$, otherwise (3) is clearly valid. But according to condition (*), this implies existence of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with

$$x\mathbf{p_1}z\mathbf{p_1}y, \ x\mathbf{p_2}y\mathbf{p_2}z, \ z\mathbf{p_3}x\mathbf{p_3}y,$$

which is a contradiction.

So, d^{B_j} satisfies (1-3) and hence corresponds to an ASWF. When $n > 2$, B_j is clearly non-dictatorial. ■

2.1 General Domains

The IP characterization obtained above can be generalized to the case in which the domain of preferences for each voter is non-identical. In general, let D be the domain of profiles over alternatives. In this case, for each set S, the d_S variables need not be well-defined for each pair of alternatives x, y, if there is no profile in which all agents in S (resp. S^c) rank x over y (resp. y over x). d_S is thus only defined for (x, y) if such profiles exist. Note that $d_S(x, y)$ is well-defined if and only if $d_{S^c}(y, x)$ is well-defined. With this proviso inequalities (1) and (2) remians valid. We only need to modify (3) to the following:

> Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,
>
> $$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) \leq 2, \quad (9)$$
>
> where the sets satisfy the following conditions (hereafter referred to as condition (**)):
>
> $$A \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in A, \text{ with } x\mathbf{p_i}z\mathbf{p_i}y,$$

$$B \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in B, \text{ with } y\mathbf{p_i}x\mathbf{p_i}z,$$
$$C \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in C, \text{ with } z\mathbf{p_i}y\mathbf{p_i}x,$$
$$U \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in U, \text{ with } x\mathbf{p_i}y\mathbf{p_i}z,$$
$$V \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in V, \text{ with } z\mathbf{p_i}x\mathbf{p_i}y,$$
$$W \neq \emptyset \text{ only if there exists } \mathbf{p_i}, i \in W, \text{ with } y\mathbf{p_i}z\mathbf{p_i}x.$$
$$\text{and } (\mathbf{p_1}, \ldots, \mathbf{p_n}) \in D.$$

The following theorem is immediate from our discussion. We omit the proof.

Theorem 4 *Every feasible integer solution to (1), (2) and (9) corresponds to an ASWF on domain D and vice-versa.*

This yields a new characterization of non-dictatorial profile domains D. As an application, we use it to prove a result on super non-Arrovian domains by Fishburn and Kelly [7].

A domain D is called **super non-Arrovian** if it is non-Arrovian and every domain D' containing D is also non-Arrovian. Furthermore, if d_S is well defined for every pair of alternatives x, y and every S, we say that the domain D satisfies the **near-free doubles** condition.

Theorem 5 (Fishburn and Kelly [7]) *A domain D is super-non-Arrovian if and only if it is non-Arrovian and satisfies the near-free doubles condition.*

Proof. Consider the case when the domain D is non-arrovian and satisfies the near-free doubles condition. Suppose $D \subseteq D'$, and D' is arrovian. Let d'_S be a solution corresponding to a non-dictatorial ASWF in the IP for domain D'. By the near-free double condition, d'_S is well-defined for every pair of alternatives for all S. Define $d_S = d'_S$. Note that since the set of constraints in the IP correspoinding to domain D is a subset of the constraints for domain D', the solution d_S is trivially a feasible solution to the IP for domain D. This contradicts the fact that domain D is non-arrovian.

Similarly, consider the case when the domain D is super non-Arrovian. It is clearly non-Arrovian. Suppose it does not satisfies the near-free double conditions, say d_S is not defined over the pair (x, y) for set S^*. Let $\mathbf{e}$ be a new voter profile where each voter in S prefers x over y, and each voter in S^c prefers y over x. All other alternatives are inferior to both x and y, but their relative orders are the same for all voters. Consider the domain $D' = D \cup \{\mathbf{P}\}$. Let d_S be a dictatorial solution (corresponding to a dictator in $(S^*)^c$) in the IP for D. Consider the IP for D'. Note that the new IP contains exactly two new variables $d_{S^*}(x, y)$ and $d_{(S^*)^c}(y, x)$. Note also that for the profile $\mathbf{P}$, any triplet involving x, y comes in only two orders: xyz and yxz for any third alternative z. Hence constraint (9) corresponding to $\mathbf{P}$ is trivially true, for any values realized by $d_{S^*}(x, y)$ and $d_{(S^*)^c}(y, x)$. Set $d_{S^*}(x, y) = 1$ and $d_{(S^*)^c}(y, x) = 0$. Append this to the solution d_S to obtain a solution for the IP for D'. This is a non-dictatorial solution for the domain D'. ∎

3 Applications

3.1 Anonymous and Neutral Rules

Two additional conditions that are sometimes imposed on an ASWF are anonymity and neutrality. An ASWF is called **anonymous** if its ranking over pairs of alternatives remains unchanged when the labels of the agents are permuted. Hence $d_S(x, y) = d_T(x, y)$ for all $(x, y) \in \mathcal{A}^2$ whenever $|S| = |T|$. In particular a dictatorial rule is not anonymous.

An ASWF is called **neutral** if its ranking over any pair of alternatives depends only on the pattern of agents' preferences over that pair, not on the alternatives' labels. Neutrality implies that $d_S(x, y) = d_S(a, b)$ for any $(x, y), (a, b) \in \mathcal{A}^2$. Thus the value of $d_S(\cdot, \cdot)$ is determined by S alone.

When anonymity and neutrality are combined, $d_S(\cdot, \cdot)$ is determined by $|S|$ alone. In such a case, we write d_S as d_r where $r = |S|$. If n is even, it is not possible for an anonymous ASWF to be neutral because Eq. (2) cannot be satisfied for $|S| = n/2$.

It is easy to see that Two obviously feasible solutions are $d_1(x, y) = 1, d_2(x, y) = 0, \forall(x, y) \in \mathcal{A}^2$ and $d_1(x, y) = 0, d_2(x, y) = 1 \forall(x, y) \in \mathcal{A}^2$. The first corresponds to an ASWF in which agent 1 is the dictator and the second, by default, to one in which agent 2 is the dictator. We refer to these solutions as the all 1's solution and all the 0's solution respectively. Majority rule is both anonymous and neutral but is not the only such rule. For three agents, the three person minority rule is anonymous and neutral. (In the three person minority rule, only singleton sets and the entire set of agents are decisive.)

Sen [17] characterizes those domains for which the majority rule is an ASWF. Maskin [11] charaterizes those domains that admit anonymous and neutral ASWF's. Here we go one step further and characterize those domains that admit a non-dictatorial, neutral ASWF. The proof uses the integer programming formulation introduced earlier. As stepping stones we need the results of Sen [17] and Maskin [11], which we also (re)derive using the IP.

Recall that Ω admits a *Condorcet* triple if there are x, y and $z \in \mathcal{A}$ and $\mathbf{p_1}$, $\mathbf{p_2}$ and $\mathbf{p_3} \in \Omega$ such that $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$. The following theorem is essentially due to Sen [17].

Theorem 6 *For an odd number of agents, majority rule is an ASWF on Ω if and only if Ω does not contain a Condorcet triple.*

Proof. Suppose first that majority rule is an ASWF on Ω. To get a contradiction assume that $x, y, z \in \mathcal{A}$ form a Condorcet triple. Let n, the number of agents, be $3r + k$ for some integers $r \geq 1$ and $k = 0, 1$, or 2.

If $k = 0$, partition the agents into three sets of size r called U, V and W. Every agent in U ranks x above y above z. Every agent in V ranks z above x above y. Every agent in W ranks y above z above x. Since n is odd and $2r > n/2$ it follows that on this profile that majority rule produces a cycle.

If $k = 1$, choose U, V and W as above but $|U| = |V| = r$ and $|W| = r + 1$. Once again $2r + 1 > 2r > n/2$, so majority rules cycles again. If $k = 2$ repeat the argument with $|U| = r$ and $|V| = |W| = r + 1$.

Now suppose that Ω has no Condorcet triple. To show that majority rule is an ASWF we must show that inequality (3) is satisfied. To obtain a contradiction suppose not and fix a triple $x, y, z \in \mathcal{A}$ for which (3) is violated. Since Ω has no Condorcet triple, at least one of A, B or C is empty and at least one of U, V and W is empty. Without loss of generality suppose that $A, W = \emptyset$. Since (3) is violated we have $d_{U\cup V}(x, y) = d_{B\cup U}(y, z) = d_{C\cup V}(z, x) = 1$. Majority rule implies that $|B|+|U| > n/2$ and $|C|+|V| > n/2$. Adding these two inequalities produces:

$$n = |B| + |U| + |C| + |V| > n,$$

a contradiction. ■

The next result we derive using the IP formulation is essentially due to Maskin [11].

Theorem 7 *Suppose there are at least 3 agents. If Ω admits an anonymous, neutral ASWF, then Ω has no Condorcet triples.*

Proof. Suppose Ω admits an anonymous, neutral ASWF f, and suppose $x, y, z \in \mathcal{A}$ is a collection that forms a Condorcet triple. Consider the d variables associated with the ASWF f. Thus d satisfies (1,2,3). Let n denote the number of agents.

Inequality (3) implies:

$$1 \leq d_a(x, y) + d_b(y, z) + d_c(z, x) \leq 2,$$

whenever

$$a, b, c > 0, a + b + c = n.$$

Note that by neutrality, $d_S(x, y) = d_S(a, b)$ for all $(x, y), (a, b)$. So we omit the alternatives and represent the variables as $d_{|S|}$.

Note that by (2), $d_1 \neq d_{n-1}$. Furthermore, since

$$1 \leq d_1 + d_1 + d_{n-2} \leq 2,$$

$d_{n-2} \neq d_1$, i.e., $d_{n-1} = d_{n-2}$. Again by (2), we must have $d_2 = d_1$. Since

$$1 \leq d_1 + d_2 + d_{n-3} \leq 2,$$

we have $d_{n-3} = d_{n-2} = d_{n-1}$. Repeating the above argument, we obtain the series of equalities:

$$d_1 = d_2 = d_3 = \ldots = d_{\lfloor \frac{n}{2} \rfloor},$$

$$d_{n-1} = d_{n-2} = \ldots = d_{\lceil \frac{n}{2} \rceil}.$$

The number of agents, n, must be odd, otherwise no anonymous ASWF is neutral. If n is odd, however,

$$d_1 + d_{\lfloor \frac{n}{2} \rfloor} + d_{\lfloor \frac{n}{2} \rfloor} = 0 \text{ or } 3,$$

a contradiction. ■

An immediate consequence of Theorems 6 and 7 is the following corollary.

Corollary 1 *Let the number of agents be odd. Then the following three statements are equivalent:*

1. *The domain Ω admits an anonymous and neutral ASWF.*
2. *Majority rule is an ASWF on Ω.*
3. *The domain Ω contains no Condorcet triple.*

Maskin [11] provides another result that shows majority rule is the most robust amongst all anonymous and neutral rules. We describe this result next.[2]

Let g be any rule that associates with each $\mathbf{P} \in \Omega^n$ a pairwise ordering of the alternatives in $\mathcal{A}$. There is no requirement that g produce a transitive ordering of the elements of $\mathcal{A}$, i.e., g need not be an ASWF.

Theorem 8 *Suppose that g is anonymous, neutral, satisfies unanimity and independence of irrelevant alternatives, and is not majority rule. Then there exists a domain Ω on which g is not an ASWF but majority rule is.*

Proof. Anonymity implies that g is not dictatorial. Thus, there has to be a domain Ω and some profile $\mathbf{P} \in \Omega^n$ on which g generates an intransitive order. Since g satisfies independence of irrelevant alternatives, we can associate a set of "decisiveness" variables with g. Call them d'.

Given that g is not a ASWF on Ω, there is a triple of alternatives $x, y, z \in \mathcal{A}$ and an appropriate partition of the agents such that constraint (3) is violated. Suppose first that Ω does not admit a Condorcet triple in the alternatives x, y, z. Let Π be the set of orderings of x, y, z admissible under Ω. Fix an ordering σ of elements of $\mathcal{A} \setminus \{x, y, z\}$. Let Ω' be the set of all preference orderings of the form (π, σ) where $\pi \in \Pi$. It is easy to see that majority rule is a ASWF on this domain but g cannot be since it would violate (3) with respect to $\{x, y, z\}$. Hence we may assume that every violation of (3) by d' on any domain is associated with a Condorcet triple.

Let a and b be two positive integers. We claim that $d'_a = d'_b = 1 \Rightarrow a + b > n$. To see why, suppose not and consider the domain consisting of the following three orderings: $\{xyz, yzx, yxz\}$. This domain does not admit a Condorcet triple and so g, equivalently d' defines an ASWF on it. Suppose now a profile where a agents have the ranking xyz, b agents

[2]See Campbell and Kelly [4] for the same result derived under slightly weaker conditions.

have the ranking yzx and the remaining $n - a - b$ agents have the ranking yxz. The first set of a agents are the only ones to rank x above y. Since $d'_a = 1$, on this domain g ranks x above y. A similar argument applies to the second set of b agents and the ordered pair zx. However, unanimity requires that g rank y above z. Hence g is not an ASWF on this domain, a contradiction.

Next we claim that if $a \leq b$ then $d'_a = 1 \Rightarrow d'_b = 1$. Suppose not. By (2) it follows that $d'_{n-b} = 1$. From the previous claim, $d'_a = d'_{n-b} = 1$ implies that $a + n - b > n$ which cannot be since $a \leq b$.

Let r be the smallest integer such that $d'_r = 1$. Suppose first that $r < n/2$. Now $d'_r = 1$ implies $d'_{r+1} = 1$. But $r+(r+1) \leq n$ a contradiction. Now assume that $r > n/2$. If $d'_{\frac{(n+1)}{2}} = 0$ we have from (2) that $d'_{\frac{(n-1)}{2}} = 1$ which contradicts the choice of r. Since $d'_{\frac{(n+1)}{2}} = 1$ it follows that $d'_a = 1$ for all $a \geq (n+1)/2$. That is g is majority rule, contradiction. ■

The next result, as far as we know, is new. It shows that checking whether Ω admits a neutral, non-dictatorial ASWF reduces to checking whether majority rule or the born loser rule is an ASWF on that domain. Notice that no parity assumption on the number of voters is needed.

Theorem 9 *For $n \geq 3$, a domain Ω admits a neutral, nondictatorial ASWF if an only if majority rule or the born loser rule is an ASWF on Ω.*

Proof. If either the majority rule or the born loser rule are ASWF's on Ω, Ω clearly admits a neutral, non-dictatorial ASWF. Suppose then Ω admits a neutral, non-dictatorial ASWF, but neither the majority rule nor the born loser rule is an ASWF on Ω. Since the majority rule is not an ASWF, Ω admits a Condorcet triple $\{a, b, c\}$. Since the born loser rule is not an ASWF on Ω, by corrollary 1 there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with

$$x\mathbf{p_1}z\mathbf{p_1}y, \ x\mathbf{p_2}y\mathbf{p_2}z, \ z\mathbf{p_3}x\mathbf{p_3}y.$$

We will need the existence of these orderings to construct a partition of the agents that satisfies the cycle elimination constraints. The proof will mimic the proof of Arrow's theorem (Theorem 2) given earlier.

Neutrality implies that $d_S(x, y) = d_S(y, z) = d_S(z, u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x, y)$ in the rest of the proof.

First, $d_S = 1 \Rightarrow d_T = 1$ for all $S \subset T$. Suppose not. Let T be the set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$ and $V = S$ in (3). We can do this because of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$. Then, $d_{A \cup U \cup V} = d_E = 1$, $d_{B \cup U \cup W} = d_{T^c} = 1$ and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \emptyset$, otherwise the assignment $A = (S \cup T)^c, U = S, V = T$ will violate the cycle elimination constraint.

Next we show that $d_{S\cap T} = 1$. Suppose not. Consider the assignment $U = E\backslash S$, $V = S\backslash T$ and $W = S \cap T$. We can choose such a partition because $\{a, b, c\}$ form a Condorcet triple. For this specification, $d_{A\cup U\cup V} = d_{E\backslash\{S\cap T\}} = 1$. Since $T \subset B \cup U \cup W$, $d_{B\cup U\cup W} = 1$ and $d_{C\cup V\cup W} = d_S = 1$, which contradicts (3).

Hence there exists a minimal set S^* such that $d_{S^*} = 1$ and all T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not there will be $j \in S$ with $d_j = 0$, and hence $d_{E\backslash\{j\}\cap S^*} = 1$, contradicting the minimality of S^*. ■

A simple consequence of this result is the following theorem due to Kalai and Muller [8]. The proof is new.

Theorem 10 *A non-dictatorial solution to (1, 2, 5 - 8) exists for the case $n = 2$ agents if and only if a non-dictatorial solution to (1-3) exists for any n.*

Proof. Given a 2 person non-dictatorial AWSF, we can build an ASWF for the n-person case by focusing only on the preferences submitted by the first two voters and rank the alternatives using the 2-person ASWF. This is clearly a non-dictatorial ASWF for the n-person case. Hence we only need to give a proof of the converse.

Let d^* be a non-dictatorial solution to (1-3). Suppose d does not imply a neutral ASWF. Then there is a set of agents S such that $d^*_S(x, y)$ is non-zero for some but not all $(x, y) \in \mathcal{A}^2$. Hence, $d_1 = d^*_S, d_2 = d^*_{S^c}$ would be a non-dictatorial solution to (1, 2, 5–8).

Suppose then d implies a neutral ASWF. By the previous theorem we can choose d to be either the majority rule or the born loser rule. In the first case, we can build a 2 person ASWF by using a dummy voter with a fixed ordering from Ω and using the (3 person) majority rule. In the second case, we can build a 2 person ASWF by adding a dummy born loser. ■

The following refinement to Maskin's result also follows directly from Theorem 9. It says that for many classes of domains, Majority Rule is essentially the only anonymous and neutral ASWF.

Theorem 11 *Let the number of agents be odd. Suppose Ω does not contain any Condorcet triples, and suppose there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with*

$$x\mathbf{p_1}z\mathbf{p_1}y,\ \ x\mathbf{p_2}y\mathbf{p_2}z,\ \ z\mathbf{p_3}x\mathbf{p_3}y.$$

Then, majority rule is the **only** *anonymous, neutral ASWF on Ω.*

Proof.(Sketch) From the proof to Theorem 9, we know that if d_S corresponds to a neutral ASWF, and if there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with $x\mathbf{p_1}z\mathbf{p_1}y$, $x\mathbf{p_2}y\mathbf{p_2}z, z\mathbf{p_3}x\mathbf{p_3}y$, then d_S is monotonic. i.e., $d_S \leq d_T$ if $S \subset T$. By May's Theorem, it has to be the majority rule since majority rule is the only ASWF that is anonymous, neutral and monotonic. ■

3.2 Single-peaked Domains

The domain Ω is single-peaked with respect to a linear ordering $\mathbf{q}$ over $\mathcal{A}$ if $\Omega \subseteq \{\mathbf{p} \in \Sigma :$ for every triple (x, y, z) if $x\mathbf{q}y\mathbf{q}z$ then it is **not** the case that $x\mathbf{p}y$ and $z\mathbf{p}y\}$. The class of single-peaked preferences has received a great deal of attention in the literature. Here we show how the IP can be used to characterize the class of ASWF's on single peaked domains. We prove that the constraints (1–3), along with the non-negative constraints on the d variables, are *sufficient* to characterize the convex hull of the 0-1 solutions.

Theorem 12 *When Ω is single-peaked the set of non-negative solutions satisfying (1-3) is an integral polytope. All ASWF's are extreme point solutions of this polytope.*

Proof. It suffices to prove that every fractional solution satisfying (1-3) can be written as a convex combination of 0-1 solutions satisfying the same set of constraints. Let $\mathbf{q}$ be the linear ordering with respect to which Ω is single-peaked.

Rounding Scheme:
Let $d_S(\cdot)$ be a (possibly) fractional solution to the linear programming relaxation of (1-3). We round the solution d to the 0-1 solution d' in the following way:

- Generate a random number Z uniformly between 0 and 1.
- For $a, b \in \mathcal{A}$ with $a\mathbf{q}b$, and $S \subset E$, then
  - $d'_S(a, b) = 1$, if $d_S(a, b) > Z$, 0 otherwise;
  - $d'_S(b, a) = 1$, if $d_S(b, a) \geq 1 - Z$, 0 otherwise.

The integral solution obtained is feasible:
The 0-1 solution d'_S generated in the above manner clearly satisfies constraints (1). To verify that it satisfies constraint (2), consider a set $T \subseteq E$, an arbitrary pair of alternatives a, b, and suppose without loss of generality $a\mathbf{q}b$. From the linear programming relaxation, we know that either $d_T(a, b) > Z$ or $d_T(b, a) \geq 1 - Z$ (since the two variables add up to 1), but not both. Thus, exactly one of $d'_T(a, b)$ or $d'_T(b, a)$ is set to 1.

We show next that all the constraints in (3) are satisfied by the solution $d'_S(\cdot)$. Consider three alternatives a, b, c, and constraint (3) (with a, b, c replacing the role of x, y, z) can be re-written as:

$$d_{A\cup U\cup V}(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup V\cup W}(c, a) \leq 2.$$

Suppose $a\mathbf{q}b\mathbf{q}c$. Then in constraints (3), by the single-peakedness property, we must have $A = V = \emptyset$. In this case, the constraint reduces to

$$d_U(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup W}(c, a) \leq 2.$$

We need to show that

$$d'_U(a,b) + d'_{B\cup U\cup W}(b,c) + d'_{C\cup W}(c,a) \le 2.$$

By choosing the sets in constraints (3) in a different way, with

$$U' \leftarrow U, B' \leftarrow B, W' \leftarrow W \cup C, C' \leftarrow \emptyset,$$

we have a new inequality

$$d_{U'}(a,b) + d_{B'\cup U'\cup W'}(b,c) + d_{C'\cup W'}(c,a) \le 2,$$

which is equivalent to

$$d_U(a,b) + 1 + d_{C\cup W}(c,a) \le 2.$$

Hence we must have $d_U(a,b) + d_{C\cup W}(c,a) \le 1$. Note that since $a\mathbf{q}b$ and $b\mathbf{q}c$, our rounding scheme ensures that $d'_U(a,b) + d'_{C\cup W}(c,a) \le 1$. Hence

$$d'_U(a,b) + d'_{B\cup U\cup W}(b,c) + d'_{C\cup W}(c,a) \le 2.$$

To finish the proof, we need to show that constraint (3) holds for different orderings of a, b and c under $\mathbf{q}$; the above argument can be easily extended to handle all these cases to show that constraint (3) is valid. We omit the details here.

<u>All extreme point solutions are integral</u>:
Suppose now that the fractional solution d_S is a fractional extreme point in the polytope defined by constraints (1-3). By standard polyhedra theory, there exists a cost function $(c_S(a,b))$ such that d_S is the unique minimum solution to the problem:

$$\begin{array}{rl} (P) \quad \min & \sum_{S,a,b} c_S(a,b)x_S(a,b) \\ \text{subject to}: & x_S(a,b) \text{ satisfies constraints (1-3);} \\ & x_S(a,b) \in [0,1]. \end{array}$$

The rounding scheme we have just described converts a fractional solution to a 0-1 solution satisfying

$$E(d'_S(a,b)) = P(Z < d_S(a,b)) = d_S(a,b), \text{ if } a\mathbf{q}b,$$

and

$$E(d'_S(a,b)) = P(Z \ge 1 - d_S(a,b)) = d_S(a,b), \text{ if } b\mathbf{q}a.$$

Hence $E(d'_S(a,b)) = d_S(a,b)$ for all S, a and b. Thus

$$E(\sum_{S,a,b} c_S(a,b)d'_S(a,b)) = \sum_{S,a,b} c_S(a,b)d_S(a,b),$$

and hence all the 0-1 solutions obtained by the rounding scheme must also be a minimum solution to problem (P). This contradicts the fact that d_S is the unique minimum solution. ■

The argument above shows the set of ASWF's on single-peaked domains (wrt $\mathbf{q}$) has a property similar to the generalized median property of the stable marriage problem (see Teo and Sethuraman [18]).

Theorem 13 *Let $f_1, f_2, \ldots, f_N$ be distinct ASWF's for the single-peaked domain Ω (with respect to $\mathbf{q}$). Define a function $F_k : \Omega^n \to \Sigma$ with the property:*

> *The set S under F_k is decisive for x over y if*
>
> > $x\mathbf{q}y$, *and S is decisive for x over y for* at least $k+1$ *of the ASWF f_i's; or*
> >
> > $y\mathbf{q}x$, *and S is decisive for x over y for* at least $N-k$ *of the ASWF f_i's.*
>
> *Then F_k is also an ASWF.*

One consequence of Theorem 13 is that when Ω is single-peaked, it is Arrovian, since the dictatorial ASWF can be used to construct non-dictatorial ASWF in the above manner. For instance, consider the case $n = 2$. Let f_1 and f_2 be the dictatorial rule associated with agent's 1 and 2 respectively. The function F_1 constructed above reduces to the following ASWF:

> If $x\mathbf{q}y$, the social welfare function ranks x above y if and only if both agents prefer x over y.
>
> If $y\mathbf{q}x$, the social welfare function ranks y above x if and only if none of the agents prefer x above y.

3.3 Generalized Majority Rule

Moulin [13] has introduced a generalization of majority rule called generalized majority rule. Generalized majority rule (GMR) M for n agents is of the following form:

- Add n-1 dummy agents, each with a fixed preference drawn from Ω.
- x is ranked above y under M if and only if the majority (of real and dummy agents) prefer x to y.

Each instance of a GMR can be described algebraically as follows. Fix a profile $\mathbf{R} \in \Omega^{n-1}$ and let $R(x, y)$ be the number of orderings in $\mathbf{R}$ where x is ranked above y. Given any profile $\mathbf{P} \in \Omega^n$, GMR ranks x above y if the number of agents who rank x above y under $\mathbf{P}$ is at least $n - R(x, y)$. To check that GMR is an ASWF on single peaked domains, set

$$g_S(x, y) = 1 \text{ iff } |S| \geq n - R(x, y)$$

and zero otherwise. It is easy to check that g satisfies (1)-(3) when Ω is single peaked.

GMR has two important properties. The first is that it is anonymous and second that it is monotonic.

Definition 2 *An ASWF is* **monotonic** *if when one switches from the profile* $\mathbf{P}$ *to* $\mathbf{Q}$ *by raising the ranking of* $x \in A$ *for at least one agent, then* $f(\mathbf{Q})$ *will not rank* x *lower than it is in* $f(\mathbf{P})$.

Theorem 14 (Moulin) *An ASWF that is anonymous and monotonic on a single-peaked domain* Ω *must be a generalized majority rule*

Proof. Let d_S be a solution to (1)-(3), corresponding to an anonymous and monotonic ASWF on the domain Ω. Let $\mathbf{q}$ be the underlying order of alternatives. For each $(x, y) \in A^2$, by anonymity, $d_S(x, y)$ depends only on the cardinality of S. Monotonicity implies $d_S(x, y) \leq d_T(x, y)$ if $S \subseteq T$. Thus

$$d_S(x, y) = 1 \text{ if and only if } |S| \geq e(x, y)$$

for some number $e(x, y)$. To complete the proof we need to determine a profile $\mathbf{R} \in \Omega^{n-1}$ such that

$$n - R(x, y) = e(x, y) \ \forall (x, y) \in A^2.$$

Since $d_S(x, y) + d_{S^c}(y, x) = 1$, we have

$$e(x, y) + e(y, x) = n + 1$$

for all (x, y) and (y, x). Note that $e(x, y) \geq 1$ and $e(x, y) \leq n$. Furthermore, if $x\mathbf{q}y\mathbf{q}z$, then by (4) and (5), $d_S(x, y) \leq d_S(x, z)$ and hence $e(x, y) \geq e(x, z)$. Similarly, we have $e(y, x) \leq e(z, x)$, $e(z, y) \geq e(z, x)$ and $e(x, z) \geq e(y, z)$.

We use the geometric construction used in the earlier proof to construct the profile $\mathbf{R} \in \Omega^{n-1}$.

- To each (x, y) such that $x\mathbf{q}y$, associate the interval $[0, e(x, y)]$ and label it $l(x, y)$.
- To each (x, y) such that $y\mathbf{q}x$, associate the interval $[n + 1 - e(x, y), n + 1]$ and label it $l(x, y)$.

We construct preferences in $\mathbf{R}$ in the following way:

- For each $k = 1, 2, \ldots, n - 1$, if $l(x, y)$ covers the point $k + 0.5$, then the kth dummy voter ranks y over x. Otherwise the dummy voter ranks x over y.

Since the intervals $l(x, y)$ and $l(y, x)$ are disjoint and cover $[0, n + 1]$ the procedure is well-defined. If $R(x, y)$ is the number of dummy voters who rank x above y in this construction it is easy to see that $n - R(x, y) = e(x, y)$, which is what we need. It remains then to to show that the profile constructed is in Ω^{n-1}.

Claim 3 *The procedure returns a linear ordering of the alternatives.*

Proof. Suppose otherwise and consider three alternatives x, y, z where the procedure (for some dummy voter) ranks x above y, y above z and z above x. Hence the intervals $l(x,y)$, $l(y,z)$ and $l(z,x)$ do not cover the point $k+0.5$. From symmetry, it suffices to consider the following two cases:

- Case 1. Suppose $x\mathbf{q}y\mathbf{q}z$. Since $l(x,z)$ covers the point $k+0.5$ and $e(x,y) \geq e(x,z)$, $l(x,y)$ must cover the point $k+0.5$, a contradiction.
- Case 2. Suppose $y\mathbf{q}x\mathbf{q}z$. Now, there exists $\mathbf{p}$ and $\mathbf{p}'$ in Ω with $z\mathbf{p}x\mathbf{p}y$ and $x\mathbf{p}'y\mathbf{p}'z$, hence $l(z,x) \geq l(z,y)$. This is impossible as $l(z,y)$ covers the point $k+0.5$ but $l(z,x)$ does not.

Hence the ordering constructed is a linear order. ■

Claim 4 *The linear orderings constructed for the dummy voters correspond to orderings from* Ω.

Proof. If not there exist k and $x\mathbf{q}y\mathbf{q}z$ with the kth dummy voter ranking y below x and z. i.e. $l(x,y)$ does not cover the point $k+0.5$ and $l(y,z)$ does. Hence $e(x,y) < e(y,z)$. Now, using $x\mathbf{q}y\mathbf{q}z$, we have

$$d_S(x,y) \leq d_S(x,z), d_S(x,z) \leq d_S(y,z).$$

So

$$e(x,y) \geq e(x,z), e(x,z) \geq e(y,z),$$

which is a contradiction. ■

3.4 Muller-Satterthwaite Theorem

A **Social Choice Function** maps profiles of preferences into a single alternative. These are objects that have received as much attention as social welfare functions. It is therefore natural to ask if the integer programming approach described above can be used to obtain results about social choice functions. Up to a point, yes. The difficulty is that knowing what alternative a social choice function will pick from a set of size two, does not, in general, allow one to infer what it will choose when the set of alternatives is extended by one. However, given the additional assumptions imposed upon a social choice function one can surmount this difficulty. We illustrate how with two examples.

The analog of Arrow's impossibility theorem for social choice functions is the Muller-Satterthwaite theorem [14]. The counterpart of Unanimity and the Independence of Irrelevant Alternatives condition for Social Choice Functions are called **pareto optimality** and **monotonicity**. To define them, denote the preference ordering of agent i in profile $\mathbf{P}$ by $\mathbf{p_i}$.

1. **Pareto Optimality:** Let $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}y$ for all $\mathbf{p} \in \mathbf{P}$. Then $f(\mathbf{P}) \neq y$.

2. **Monotonicity:** For all $x \in \mathcal{A}$, $\mathbf{P}, \mathbf{Q} \in \Omega^n$ if $x = f(\mathbf{P})$ and $\{y : x\mathbf{p_i}y\} \subseteq \{y : x\mathbf{q_i}y\}$ $\forall i$ then $x = f(\mathbf{Q})$.

We call a Social Choice Function that satisfies pareto-optimality and monotonicity an Arrovian Social Choice Function (ASCF).

Theorem 15 (Muller-Satterthwaite) *When $\Omega = \Sigma$, all ASCF's are dictatorial.*[3]

Proof: For each subset S of agents and ordered pair of alternatives (x, y), denote by $[S, x, y]$ the set of all profiles where agents in S rank x first and y second, and agents in S^c rank y first and x second. By the hypothesis on Ω this collection is well defined.

For any profile $\mathbf{P} \in [S, x, y]$ it follows by pareto optimality that $f(\mathbf{P}) \in \{x, y\}$. By monotonicity, if $f(\mathbf{P}) = x$ for one such profile $\mathbf{P}$ then $f(\mathbf{P}) = x$ for all $\mathbf{P} \in [S, x, y]$.

Suppose then for all $\mathbf{P} \in [S, x, y]$ we have $f(\mathbf{P}) \neq y$. Let $\mathbf{Q}$ be any profile where all agents in S rank x above y, and all agents in S^c rank y above x. We show next that $f(\mathbf{Q}) \neq y$ too.

Suppose not. That is $f(\mathbf{Q}) = y$. Let $\mathbf{Q}'$ be a profile obtained by moving x and y to the top in every agents ordering but preserving their relative position within each ordering. So, if x was above y in the ordering under $\mathbf{Q}$, it remains so under $\mathbf{Q}'$. Similarly if y was above x. By monotonicity $f(\mathbf{Q}') = y$. But monotonicity with respect to $\mathbf{Q}'$ and $\mathbf{P} \in [S, x, y]$ implies that $f(\mathbf{P}) = y$ a contradiction.

Hence, if there is one profile in which all agents in S rank x above y, and all agents in S^c rank y above x, and y is not selected, then all profiles with such a property will not select y. This observation allows us to describe ASCF's using the following variables.

For each $(x, y) \in \mathcal{A}^2$ define a 0-1 variable as follows:

- $g_S(x, y) = 1$ if when all agents in S rank x above y and all agents in S^c rank y above x then y is never selected,
- $g_S(x, y) = 0$ otherwise.

If E is the set of all candidates we set $g_E(x, y) = 1$ for all $(x, y) \in \mathcal{A}^2$. This ensures pareto optimality.

Consider a $\mathbf{P} \in \Omega^n$, $(x, y) \in \mathcal{A}^2$ and subset S of agents such that all agents in S prefer x to y and all agents in S^c prefer y to x. Then, $g_S(x, y) = 0$ implies that $g_{S^c}(y, x) = 1$ to ensure a selection. Hence for all S and $(x, y) \in \mathcal{A}^2$ we have

$$g_S(x, y) + g_{S^c}(y, x) = 1. \tag{10}$$

We show that the variables g_S satisfy the cycle elimination constraints. If not there exists a triple $\{x, y, z\}$, and set A, B, C, U, V, W such that the cycle elimination constraint is violated. Consider the profile $\mathbf{P}$ where each voter ranks the triple $\{x, y, z\}$ above the rest,

[3]The more well known result about strategy proof social choice functions is due to Gibbard [6] and Satterthwiate [16]. It is a consequence of Muller-Satterthwaite [14].

and with the ordering of x, y, z depending on whether the voter is in A, B, C, U, V or W. Since $g_{A \cup U \cup V}(x, y) = 1$, $g_{B \cup U \cup W} = 1$, and $g_{C \cup V \cup W} = 1$, none of the alternatives x, y, z is selected for the profile $\mathbf{P}$. This violates pareto optimality, a contradiction.

Hence g_S satisfies constraints (1-3). Since $\Omega = \Sigma$, by Arrow's Impossibility Theorem, g_S corresponds to a dictatorial solution. ■

3.5 Strategic Candidacy

One of the features of the classical social choice literature is that the alternatives are assumed as given. Yet there are many cases where the alternatives under consideration are generated by the participants. A recent paper that considers this issue is Dutta, Jackson and Le Breton [5]. In their paper the alternatives correspond to the set of candidates for election. Candidates can withdraw themselves and in so doing affect the outcome of an election. Under some election rules (and an appropriate profile of preferences) a losing candidate would be better off withdrawing to change the outcome. They ask what election rules would be immune to such manipulations. Here we show how the integer programming approach can be used to derive the main result of their paper.

Let N be the set of voters and $\mathcal{C}$ the set of candidates, of which there are at least three. (In this section, we use "voters" instead of "agents" and "candidates" instead of "alternatives" to be consistent with the notation of [5]) For brevity we shall assume that $N \cap \mathcal{C} = \emptyset$ but it is an assumption that can be dropped. Voters have (strict) preferences over elements of $\mathcal{C}$ and the preference domain is the set of all orderings on $\mathcal{C}$ (i.e. $\Omega = \Sigma$). Candidates also have preferences over other candidates, however, each candidate must rank themselves first. There is no restriction on how they order the other candidates.

For any $\mathcal{A} \subset \mathcal{C}$ and profile $\mathbf{P}$ a voting rule is a function $f(\mathcal{A}, \mathbf{P})$ that selects an element of $\mathcal{A}$. Dutta et al. impose four conditions on the voting rule f:

1. For any $\mathcal{A} \subset \mathcal{C}$ and any two profiles $\mathbf{P}$ and $\mathbf{Q}$ which coincide on $\mathcal{A}$ we have

$$f(\mathcal{A}, \mathbf{P}) = f(\mathcal{A}, \mathbf{Q}).$$

2. If $\mathbf{P}$ and $\mathbf{Q}$ are two profiles that coincide on the set of voters N, then

$$f(\mathcal{A}, \mathbf{P}) = f(\mathcal{A}, \mathbf{Q}).$$

   Hence the outcome of the voting procedure does not depend on the preferences of the candidate.

3. **Unanimity**: Let $a \in \mathcal{A} \subset \mathcal{C}$ and consider a profile $\mathbf{P}$ where a is the top choice for all voters in the set $\mathcal{A}$, then $f(\mathcal{A}, \mathbf{P}) = a$.

4. **Candidate Stability**: For every profile $\mathbf{P}$ and candidate $c \in \mathcal{C}$ we have that candidate c prefers $f(\mathcal{C}, \mathbf{P})$ to $f(\mathcal{C} \setminus c, \mathbf{P})$.

A voting rule is dictatorial if there is a voter $i \in N$ such that for all profiles $\mathbf{P}$ and all $c \in \mathcal{C}$, $f(\mathcal{C}, \mathbf{P})$ and $f(\mathcal{C} \setminus c, \mathbf{P})$ are voter i's top ranked choices. This is a dictatorship condition weaker than usually considered. The main result of Dutta, Jackson and Le Breton [5] is that the *the only voting rules that satisfy 1, 2, 3 and 4 are dictatorial.*

Claim 5 *If f is a voting rule, $f(\mathcal{C}, \mathbf{P}) = a$, and $c \neq a$, then $f(\mathcal{C} \setminus c, \mathbf{P}) = a$.*

Proof. If not there exists a candidate c whose exit will change the outcome from a to $f(\mathcal{C} \setminus c, \mathbf{P})$. Since the outcome of the voting procedure does not depend on the preferences of the candidate, consider the situation where c ranks candidate a as the least preferred alternative. In this instance, c benefits by exiting the election. This contradicts the assumption that f is candidate stable (cf. Condition 4). ■

Claim 6 *If f is a voting rule, and $\mathbf{P}$ a profile in which every voter ranks all candidates in the set $\mathcal{A}$ above all candidates in the set $\mathcal{A}^c$, then $f(\mathcal{C}, \mathbf{P}) \in \mathcal{A}$.*

Proof. The statement is true by unanimity if $|\mathcal{A}| = 1$. If $|\mathcal{A}| = k + 1$, and $f(\mathcal{C}, \mathbf{P}) \notin \mathcal{A}$, then by candidate stability, $f(\mathcal{C} \setminus c, \mathbf{P}) \notin \mathcal{A}$ for $c \in \mathcal{A}$. Let $\mathbf{Q}$ be a profile obtained from P by putting c as the least preferred candidate for all voters. Then in $\mathbf{Q}$, all voters prefer the $|k|$ candidates in $\mathcal{A} \setminus c$ to the rest. Hence by induction, $f(\mathcal{C}, \mathbf{Q}) \in \mathcal{A} \setminus c$. By candidate stability again, $f(\mathcal{C} \setminus c, \mathbf{Q}) \in \mathcal{A} \setminus c$. By condition 1, $f(\mathcal{C} \setminus c, \mathbf{P}) = f(\mathcal{C} \setminus c, \mathbf{Q})$, which is a contradiction. ■

Let $[S, x, y]$ denote the set of profiles such that all voters in $S \subset N$ rank x and y, and all agents in S^c rank y and x as their top two alternatives (in that order).

Claim 7 *For all profiles $\mathbf{P}, \mathbf{Q} \in [S, x, y]$, $f(\mathcal{C}, \mathbf{P}) = f(\mathcal{C}, \mathbf{Q})$.*

Proof. We need to show that for all $P \in [S, x, y]$, $f(\mathcal{C}, \mathbf{P})$ depends only on the set S.

Without loss of generality, label the candidates as $x_1, x_2, \ldots, x_n$, with $x = x_1, y = x_2$. Let $T(k)$ be the set of all profiles where $x_1, \ldots, x_k$ are ranked above $x_{k+1}, \ldots, x_n$. Furthermore, $x_{k+1}, \ldots, x_n$ are ranked in that order by all voters. By definition, $T(n)$ is the set of all orderings over $\mathcal{C}$.

Note that there is a unique $\mathbf{P}^* \in T(2) \cap [S, x, y]$. Suppose for all $\mathbf{P} \in T(k) \cap [S, x, y]$, $f(\mathcal{C}, \mathbf{P}) = f(\mathcal{C}, \mathbf{P}^*)$. Consider $\mathbf{Q} \in (T(k+1) \setminus T(k)) \cap [S, x, y]$ and suppose $f(\mathcal{C}, \mathbf{Q}) \neq f(\mathcal{C}, \mathbf{P}^*)$. By the previous claim, since $Q \in [S, x, y]$, $f(\mathcal{C}, \mathbf{Q}) \in \{x_1, x_2\}$.

Suppose $f(\mathcal{C}, \mathbf{Q}) = x_1$, and $f(\mathcal{C}, \mathbf{P}^*) = x_2$. By candidate stability, $f(\mathcal{C} \setminus x_{k+1}, \mathbf{Q}) = x_1$, and $f(\mathcal{C} \setminus x_{k+1}, \mathbf{P}^*) = x_2$. Construct a new profile $\mathbf{R}$ from $\mathbf{Q}$ by moving the candidate x_{k+1} to the bottom of every voter's list. By condition 1, $f(\mathcal{C} \setminus x_{k+1}, \mathbf{R}) = f(\mathcal{C} \setminus x_{k+1}, \mathbf{Q}) = x_1$. Now, $\mathbf{R} \in T(k) \cap [S, x, y]$, hence $f(\mathcal{C}, \mathbf{R}) = f(\mathcal{C}, \mathbf{P}^*)$. By candidate stability again, we have $f(\mathcal{C} \setminus x_{k+1}, \mathbf{R}) = f(\mathcal{C}, \mathbf{P}^*) = x_2$. This is a contradiction. ■

Definition 3 *Let $d_S(x, y) = 1$ iff there is a profile $\mathbf{P} \in [S, x, y]$ where $f(\mathcal{C}, \mathbf{P}) = x$.*

Claim 8 *Constraint (3) is a valid inequality for the variables d_S defined above.*

Proof. Suppose not. Then there exist triplets x, y, z and sets A, B, C, U, V, W in condition (*) such that

$$d_{A \cup U \cup V}(x, y) = d_{B \cup U \cup W}(y, z) = d_{C \cup V \cup W}(z, x) = 1.$$

WLOG, let $x_1 = x, x_2 = y, x_3 = z$ and let $\mathbf{P}$ be a profile in $T(3)$, where the preferences of the voters are given by the sets A, B, C, U, V, W. Since $f(C, \mathbf{P}) \in \{x, y, z\}$, we may assume, for ease of exposition, that $f(C, \mathbf{P}) = x$. By candidate stability, $f(C \setminus y, \mathbf{P}) = f(C \setminus z, \mathbf{P}) = x$. Let $\mathbf{P_y}$ and $\mathbf{P_z}$ be the profiles obtained from $\mathbf{P}$ by moving y and z to the bottom of all voters' list respectively. By condition 1, $f(C \setminus y, \mathbf{P_y}) = f(C \setminus z, \mathbf{P_z}) = x$. Note that for the profile $\mathbf{P_y}$, the set of voters in $C \cup V \cup W$ rank z above x and all the other voters rank x above z. By assumption, $d_{C \cup V \cup W}(z, x) = 1$, hence $f(C, \mathbf{P_y}) = z$. By candidate stability, $f(C \setminus y, \mathbf{P_y}) = f(C, \mathbf{P_y}) = z$, a contradiction. Hence constraint (3) must be valid for the variables d_S defined above. ∎

It is also easy to see that the variables d_S satisfy equations (1) and (2). We know that when all possible orderings of C are permitted, the only solution to (1), (2) and (3) is of the following kind: there is a voter i such that for all x, $y \in C$ and all $S \ni i$ we have $d_S(x, y) = 1$. It remains to show that this implies dictatorship in the sense defined earlier. This can be done via induction over $T(k)$ as in claim 5. Hence the only voting rules that satisfy conditions 1,2, 3 and 4 are the dictatorial ones.

4 Decomposability, Complexity and Valid Inequalities

A domain is called **decomposable** if and only if there is a non-trivial solution (not all 1's or all 0's) to the system of inequalities (1, 2, 5–8) for the case $n = 2$. The main result of [8] (cf. Theorem 10) can be phrased as follows: *the domain Ω is non-dictatorial if and only if it is decomposable.* This result allows one to formulate the problem of deciding whether Ω is arrovian as an integer program involving a number of variables and constraints that is polynomial in $|\mathcal{A}|$. However, the set $\mathcal{A}$ is not the only input to the problem. The preference domain Ω is also an input. If Ω is specified by the set of permutations it contains, and if it has exponentially many permutations (say $O(2^{|\mathcal{A}|})$, the the straight forward input model needs at least $O(2^{|\mathcal{A}|})$ bits. Recall the number of decision variables for the integer program for 2-person ASWF's is polynomial in $|\mathcal{A}|$. Furthermore, the time complexity of verifying the existence of triplets in Ω can trivially be performed in time $O(n^3 2^{|\mathcal{A}|})$. Hence the decision version of the decomposability conditions can be solved in time polynomial in the size of the input.

Suppose, however, instead of listing the elements of Ω, we prescribe a polynomial time oracle to check membership in Ω. The complexity issue of deciding whether the domain is decomposable now depends on how we encode the membership oracle, and not on the number of elements in Ω. In this model, we exhibit an example to show that checking whether a triplet exists in Ω is already NP-hard.

Let G be a graph with vertex set V. Let Ω_G consist of all orderings of V that correspond to a Hamiltonian path in G. Given any triple $(u, v, w) \in V$, the problem of deciding if G admits a Hamiltonian path in which u preceedes v preceedes w is NP-complete.[4] Hence the problem of deciding whether there is a preference ordering $\mathbf{p}$ in Ω with $u\mathbf{p}v\mathbf{p}w$ is already NP-complete.

Thus, given an Ω specified by hamiltonian paths, it is already NP-hard just to write down the set of inequalities specified by the decomposability conditions!

One way to by-pass the above difficulties is to focus on odering on triplets that are realised by some preferences in Ω. The input to the complexity question is thus the set of orderings on triplets ($O(n^3)$ size) that are admissable in Ω. We will focus on this input model for the rest of the paper.

Ignore, for the moment, inequalities of types (7) and (8). The constraint matrix associated with the inequalities of types (1, 2, 5, 6) and $0 \leq d(x, y) \leq 1 \; \forall (x, y) \in \mathcal{A}^2$ is totally unimodular. This is because each inequality can be reduced to one that contains at most two coefficients of opposite sign and absolute value of 1.[5] Hence the extreme points are all 0-1. If one or more of these extreme points was different from the all 0's solution and all 1's solution we would know that Ω is Arrovian. If the only extreme points were the all 0's solution and all 1's solution that would imply that Ω is not Arrovian.

Thus difficulties with determining the existence of a feasible 0-1 solution different from the all 0's and all 1's solution have to do with the inequalities of the form (7) and (8). Notice that any admissible ordering (by Ω) of three alternatives gives rise to an inequality of types (7) and (8). However some of them will be redundant. Constraints (7, 8) are not redundant only when they are obtained from a triplet (x, y, z) with the property:

There exists $\mathbf{p}$ such that $x\mathbf{p}y\mathbf{p}z$ but no $\mathbf{q} \in \Omega$ such that $y\mathbf{q}z\mathbf{q}x$ or $z\mathbf{q}x\mathbf{q}y$.

Such a triplet is called an *isolated triplet.*

Call the inequality representation of Ω, by inequalities of types (1, 2, 5, 6), the **unimodular representation** of Ω. Note that all inequalities in the unimodular representation are of the type $d(x, u) \leq d(x, v)$ or $d(u, x) \leq d(v, x)$. *Furthermore,* $d(x, u) \leq d(x, v)$ *and* $d(u, y) \leq d(v, y)$ *appear in the representation only if there exist* $\mathbf{p}, \mathbf{q}$ *with* $u\mathbf{p}x$ *and* $v\mathbf{p}x$ and $x\mathbf{q}u$ *and* $x\mathbf{q}v$.

This connection allows us to provide a graph-theoretic representation of the unimodular representation of Ω as well as a graph-theoretic interpretation of when Ω is not Arrovian.

With each *non-trivial* element of $\mathcal{A}^2$ we associate a vertex. If in the unimodular representation of Ω there is an inequality of the form $d_1(a, b) \leq d_1(x, y)$ where (a, b) and $(x, y) \in \mathcal{A}^2$ then insert a *directed* edge from (a, b) to (x, y). Call the resulting directed graph D^Ω.

[4] If not, we can apply the algorithm for this problem thrice to decide if G admits a Hamiltonian cycle.
[5] It is well known that such matrices are totally unimodular. See for example, Theorem 11.12 in [1].

If (x, y) is a trivial pair (and hence $(x, y) \notin D^\Omega$), then $d_1(x, y)$ is automatically fixed at 1, and $d_1(y, x)$ fixed at 0. An inequality of the form $d_1(x, y) \leq d_1(x, z)$ (or $d_1(z, y)$) *cannot* appear in the unimodular representation, for any alternative z in $\mathcal{A}$. Otherwise there must be some $\mathbf{p} \in \Omega$ with $y\mathbf{p}x$. Similarly, if (x, y) is trivial, $d_1(y, x) \geq d_1(z, x)$ (or $d_1(y, z)$) *cannot* appear in the unimodular representation, for any alternative z in $\mathcal{A}$. Thus fixing the values of $d_1(x, y)$ and $d_1(y, x)$ arising from a trivial pair (x, y) does not affect the value of $d_1(a, b)$ for $(a, b) \in D^\Omega$.

A subset S of vertices in D^Ω is **closed** if there is no edge directed out of S. That is, there is no directed edge with its tail incident to a vertex in S and its head incident to a vertex outside S. Notice that $d_1(x, y) = 1 \ \forall (x, y) \in S$ and 0 otherwise (and together with those arising from the trivial pairs) is a feasible 0-1 solution to the unimodular representation of Ω if S is closed. Hence every closed set in D^Ω corresponds to a feasible 0-1 solution to the unimodular representation. The converse is also true.

Theorem 16 *If D^Ω is strongly connected then Ω is non-Arrovian.*

Proof. The set of all vertices of D^Ω is clearly a closed set. The solution corresponding to this closed set is the ASWF where agent 1 is the dictator. The empty set of vertices is closed and this corresponds to agent 2 being the dictator. If D^Ω is strongly connected,[6] these are the only closed sets in the graph. Since any ASWF must correspond to some closed set in D^Ω, we conclude that Ω is non-Arrovian. ■

We note that verifying whether a directed graph is strongly connected can be done efficiently. See [1] for details. Note also that if Ω does not contain any isolated triplets, then Ω is Arrovian if and only D^Ω is not strongly connected.

The impossibility results of Kalai, Muller and Satterthwiate [9] (saturated domains) as well as Aswal and Sen [3] (linked domains) follow directly from this result.

To understand the complexity of the problem further, we examine the polyhedral structure of the IP formulation when the number of alternatives is small. Note that checking membership of a triplet is now easy since the number of possible orderings is small.

We describe a sequential lifting method to derive valid inequalities for the problem to strengthen the LP formulation, using the directed graph D^Ω defined previously.

We say that the node u *dominates* the node v if there is a directed path in D^Ω from v to u (i.e. $d(u) \geq d(v)$).

Sequential Lifting Method:

- For each isolated triplet (x, y, z), we have the inequality

$$1 + d(x, z) \geq d(x, y) + d(y, z). \tag{11}$$

[6] A directed graph is strongly connected if there is a directed cycle through every pair of vertices.

- Let $D(x,y)$ (and resp. $D(y,z)$) denote the set of nodes in D^Ω that are dominated by the node (x,y) (resp. (y,z)) in D^Ω.
- For each node (a,b) in D^Ω, if

  $$u \in D(a,b) \cap D(x,y) \neq \emptyset, \;\; v \in D(a,b) \cap D(y,z) \neq \emptyset,$$

  then the constraint arising from the isolated triplet can be augmented by the following valid inequalities:

  $$d(a,b) + d(x,z) \geq d(u) + d(v). \tag{12}$$

To see the validity of the above constraint, note that by the definition of domination, we have

$$d(x,y) \geq d(u), d(y,z) \geq d(v), d(a,b) \geq d(u), d(a,b) \geq d(v).$$

If $d(a,b) = 0$, then $d(u) = d(v) = 0$ and hence (12) is trivially true. If $d(a,b) = 1$, then (12) follows from (11).

4.1 Example with Three Alternatives

We first show that the polyhedron defined by (1, 2, 5-8) need not be integral using a simple example. Let $\mathcal{A} = \{x, y, z\}$, and let

$$\Omega = \{xyz, yzx, zxy, xzy\}.$$

From (5–8), we get the following system of inequalities:

$$d(x,y) \leq d(x,z),$$
$$d(z,x) \leq d(y,x),$$
$$d(y,z) \leq d(y,x),$$
$$d(x,y) \leq d(z,y),$$
$$d(z,x) \leq d(z,y),$$
$$d(y,z) \leq d(x,z),$$
$$d(x,z) + d(z,y) \leq 1 + d(x,y),$$
$$d(y,z) + d(z,x) \geq d(y,x).$$

A fractional extreme point of this system is

$$d(x,z) = d(y,x) = d(y,z) = d(z,x) = d(z,y) = 0.5; \;\; d(x,y) = 0.$$

The only other fractional extreme point is:

$$d(x,y) = d(x,z) = d(y,z) = d(z,x) = d(z,y) = 0.5; \;\; d(y,x) = 1.$$

We next use the sequential lifting method to identify new valid inequalities from the isolated triplet (x, z, y).

Consider the following set of inequalities:

$$1 + d(x, y) \geq d(x, z) + d(z, y) \tag{13}$$

Note that (x, z) dominates (y, z), and (z, y) dominates (z, x). We also have $d(y, x) \geq d(y, z)$, and $d(y, x) \geq d(z, x)$, and hence (y, x) dominates both (y, z) and (z, x). The sequential lifting method gives rise to

$$d(x, y) + d(y, x) \geq d(y, z) + d(z, x). \tag{14}$$

Also, for a pair of alternatives a and b, replacing $d(a, b)$ with $1 - d(b, a)$, results in another valid inequality, which we record as

$$d(x, y) + d(y, x) \leq d(z, y) + d(x, z). \tag{15}$$

More importantly, equations (14) and (15) are *facets*. To see this, we first observe that the underlying polyhedron is full-dimensional (dimension 6); and its extreme points are

$$\Big\{e_4 = (0, 1, 0, 0, 0, 0),\ e_5 = (0, 1, 1, 1, 0, 0),\ e_6 = (1, 1, 0, 0, 0, 1),$$
$$e_7 = (1, 1, 1, 0, 1, 1),\ e_8 = (1, 1, 1, 1, 0, 1),\ e_9 = (1, 1, 1, 1, 1, 1)\Big\},$$

where the components of each entry represent $d(x, y)$, $d(x, z)$, $d(y, x)$, $d(y, z)$, $d(z, x)$, and $d(z, y)$ (in that order). The elements e_1, e_2, e_3, e_4, e_5 and e_9 are affinely independent, and satisfy (14) as an equality; the elements e_1, e_3, e_5, e_7, e_8, and e_9 are affinely independent, and satisfy (15) as an equality. These two observations show, respectively, that equations (14) and (15) are facets.

For $|\mathcal{A}| = 3$, we enumerate all possible domains, and observe that the strengthened formulation using the sequential lifting method defines the convex hull of all ASWF's in each case.

4.2 Examples with Four Alternatives

Preliminary observations. For $|\mathcal{A}| = 4$, the number of possible domains, Ω, is 2^{24}; of these, we ignore domains that contain at most 1 alternative, which leaves us $2^{24} - 25$ possibilities to consider. From the IP formulation, however, we know that two domains that generate the same set of "triplets," are either both dictatorial or both non-dictatorial; so the number of possibilities to be examined depends only on the number of different sets of triplets generated by the domains. This observation reduces the number of possibilities to 77850, which is substantially smaller than all potential subsets of triples. In addition, we can further restrict the possibilities to be explored using "symmetries:" Two distinct collections of triplets are isomorphic if one collection can be obtained from the other by simply renaming the alternatives. Clearly, if a collection of triplets (generated by some domain) is dictatorial, so are all of its isomorphic equivalents. The number of distinct collections of triplets that are not isomorphic to one another is 3315.

LP/IP relationship. For $|\mathcal{A}| = 3$, we observed that whenever the LP relaxation of the original formulation (1, 2, 5–8) has a non-trivial (possibly fractional) solution, so does the corresponding IP. This raises the possibility that although finding a compact description of the set of all SWFs may be difficult, or even impossible, we might still be able to resolve the existence/non-existence of an SWF by solving the associated LP relaxation; such a result is true for the stable roommates problem; see [18]. The following example, however, rules out such a possibility.

Example 1: Let

$$\Omega = \{wyzx, wzxy, xywz, xzyw, ywzx, yzxw, zwxy\}.$$

The associated set of triplets is

$$\begin{aligned} T = \{&wxy, wyx, xyw, ywx, yxw, wzx, xwz, xzw, zwx, zxw \\ &wyz, wzy, ywz, yzw, zwy, zyw, xyz, xzy, yzx, zxy\} \end{aligned}$$

It is easily verified that all the decision variables except $d(x, y)$ and $d(y, x)$ are equal to one another in every feasible LP solution; this is a consequence of the basic formulation (1, 2, 5–8). The fractional LP solution

$$d(x, y) = 0, \; d(y, x) = 1; \quad \text{all other variables} = 0.5,$$

is feasible. If the variables are restricted to be 0-1, it is easy to verify that Ω is dictatorial.

Additional valid inequalities. Consider the domain

$$\Omega = \{wxyz, wxzy, wzyx, zwyx, zxyw, zywx\},$$

with the associated set of triplets being

$$\begin{aligned} T = \{&wxy, xyw, ywx, wyx, zwx, wxz, wzx, zxw, \\ &wzy, zyw, zwy, wyz, zxy, xyz, xzy, zyx\} \end{aligned}$$

In T, the triplet wyx is the only isolated triplet; if it were absent, the LP relaxation of the IP associated with T would be exact.

As before, we try to strengthen the formulation by finding additional valid inequalities using the sequential lifting method. To that end, consider the following set of inequalities, each of whose justification is included alongside, in parenthesis:

$$
\begin{aligned}
d(w,x)+1 &\geq d(y,x)+d(w,y), \quad \text{(isolated triplet } wyx) && (16)\\
d(w,y) &\geq d(x,y), \quad \text{(by } \{ywx, wxy\}) && (17)\\
d(z,x) &\geq d(z,w), \quad \text{(by } \{zwx, wxz\}) && (18)\\
d(z,w) &\geq d(x,w), \quad \text{(by } \{wzx, zxw\}) && (19)\\
d(x,w) &\geq d(x,y), \quad \text{(by } \{xyw, ywx\}) && (20)\\
d(z,x) &\geq d(y,x). \quad \text{(by } \{xzy, zyx\}) && (21)
\end{aligned}
$$

From Eqs. (18)-(20), we have

$$d(z,x) \geq d(x,y). \tag{22}$$

From Eqs. (16) and (17), we have

$$d(w,x)+1 \geq d(y,x)+d(x,y), \tag{23}$$

which together with Eqs. (21) and (22) imply

$$d(w,x)+d(z,x) \geq d(y,x)+d(x,y). \tag{24}$$

As before, replacing $d(a,b)$ with $1-d(b,a)$, results in another valid inequality, which we record as

$$d(x,w)+d(x,z) \leq d(x,y)+d(y,x). \tag{25}$$

It is easy, but tedious, to verify that inequalities (24) and (25) are facets of the underlying polyhedron. It is also interesting to note that these are the first inequalities involving *four* alternatives.

When $|\mathcal{A}|=4$, we observe that, for each domain, the LP formulation, augmented with inequalities constructed using the sequential lifting method, whenever applicable, defines the convex hull of all ASWF's. A natural question is if whether the sequential lifting method will gives rise to all facets even for the case $|\mathcal{A}| \geq 5$; we do not yet know, although we suspect the answer to be negative.

5 Conclusion

In this paper, we study the connection between Arrow's Impossibility Theorem and Integer Programming. We show that the set of ASWF's can be expressed as integer solutions to a system of linear inequalities. Many of the well known results connected to the impossibility theorem are direct consequences of the Integer Program. Furthermore, the polyhedral structure of the IP formulation warrants further study in its own right. We have initiated the study on this class of polyhedra by characterizing the polyhedral structure of ASWF's on single peaked domain. We have also demonstrated by an extensive computational experiment that the sequential lifting method proposed in this paper can be used to obtain the complete polyhedral description of ASWF's when the number of alternatives is small. Several interesting problems still remain:

1. Given a domain Ω specified by certain membership oracle, is it possible to check for existence of non-dictatorial ASWF's in polynomial time? Is the problem in the class NP?

2. The LP relaxation of our proposed IP formulation characterizes the ASWF's for single peaked domain. What are the domains that can be characterized by the LP relaxation given by the sequential lifting method?

3. Can the conditions for ASCF's be written down as a system of integer linear inequalities?

We leave the above questions for future research.

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