3.1 Polytope $\implies$ bounded polyhedron

Last lecture, we were attempting to prove the Minkowsky-Weyl Theorem: every polytope is a bounded polyhedron, and every bounded polyhedron is a polytope. The second direction (every bounded polyhedron is a polytope) was shown last lecture, using an argument about corner points. This lecture, we show that every polytope is a bounded polyhedron by investigating the concept of polar duality.

**Theorem 3.1.** $\text{CONVEXHULL}(w_1, \ldots, w_k) = P = \text{Polytope} \implies P$ is a bounded polyhedron

Assume that $P$ is full-dimensional, and, WLOG, 0 is in the interior of $P$. (This latter requirement can simply be seen as a normalization, since it is easily accomplished by translation.) From this, we can deduce that there must be some ball that fits inside $P$:

$$\exists r > 0 \text{ s.t. } B(0, r) \subset P$$

We now define the polar dual of a polytope $P$, denoted $P^*$:

**Definition 3.1.** The polar dual of a set $P$, denoted $P^*$, is the set $\{y \mid y^T x \leq 1, \forall x \in P\}$.

When $P$ is a polytope, as in this case, the following definition is equivalent:

$$P^* = \{y \mid y^T w_i \leq 1, i = 1 \ldots k\}$$

It is easy to see that these two definitions are equivalent, because $\forall x \in P$, $x = \lambda_1 w_1 + \ldots + \lambda_k w_k, \lambda_i \geq 0$. Therefore, $y^T x = \sum_{i=1}^{k} \lambda_i y^T \cdot w_i \leq \sum_{i=1}^{k} \lambda_i = 1$.

**Lemma 3.2.** $P^*$ is a bounded polyhedron.

**Proof.** $r \frac{y}{\|y\|} \in B(0, r) \subset P$. So $r \frac{y}{\|y\|} \in P$. Thus, $y^T \left(\frac{r y}{\|y\|}\right) \leq 1$. Simplifying, $\frac{\|y\|^2}{\|y\|} r \leq 1$, so $\|y\| \leq \frac{1}{r}$. Since the length of $y$ is bounded, $P^*$ is a bounded polyhedron.
We still need to show that \((P^*)^*\) is a bounded polyhedron and \(P^{**} = P\). Once we’ve proven that, we’ve proven that any polytope \(P\) is a bounded polyhedron, so we’re done.

The following argument is tempting, but wrong. Note that:

\[
\forall x \in P, \forall y \in P^* x \cdot y \leq 1
\]

Flipping this around, we see:

\[
\forall y \in P^*, \forall x \in P x \cdot y \leq 1
\]

This looks a lot like the requirement for membership in \(P^{**}\)! Unfortunately, this intuition is wrong if \(P\) isn’t convex. It is possible to find \(S = \{w_1, w_2, \ldots, w_k\}\) such that \(S^* = P^*\), so \(S^{**} = P^{**} = P \neq S!\) (When \(S\) is non-convex.)

Here’s an alternate approach that does work: we show that \(P \subset P^{**}\) and \(P^{**} \subset P\).

**Proof.** The first direction is easy: consider \(x \in P\). For any \(y \in P^*\), \(x \cdot y \leq 1\). Therefore, \(x \in P^{**}\) as well, since the only requirement is that \(y \cdot x \leq 1\), which was already ensured.

For the second direction, we wish to show that for \(x \notin P\), \(x \notin P^{**}\). Let \(C, \delta\) define a hyperplane separating the polytope from \(x\): \(C^T \cdot z < \delta \forall z \in P\), and \(C^T \cdot x > \delta\). Since \(0 \in P\), \(C^T \cdot 0 < \delta\), implying that \(\delta > 0\). WLOG, let \(\delta = 1\).

We have: \(C^T \cdot z < 1 \forall z \in P\). So by the definition of \(P^*\), \(C \in P^*\). Since \(C^T x > 1\) and \(C \in P^*\), \(x \notin P^{**}\). \(\Box\)

### 3.2 Homework

#### 3.2.1 Homework #1

Construct polynomial time algorithms for the following:

1. A simple polygon is one that has no self-intersections (two edges that cross) or self-touching (an edge that passes through a vertex, or two vertices with the same coordinate). Given an ordered list of points, \(p_1, p_2, \ldots, p_n\), determine if the polygon they define, \(\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)\) is simple.

2. Given a simple polygon \(\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)\), determine if a point \(z\) is in the polygon or not.

3. Find a triangulation of a simple polygon, \(\text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)\). A triangulation is a set of triangles, \(T_1, T_2, \ldots, T_k\) such that:

   (a) \(T_1 \cup T_2 \cup \ldots \cup T_k = \text{POLYGON}(p_1, p_2, \ldots, p_n, p_1)\)

   (b) \(\text{interior}(T_i) \cap \text{interior}(T_j) = \emptyset, \forall i \neq j\)

   (c) \(\forall \text{vertices}(T_i) \subset p_1, p_2, \ldots, p_n\)
3.2.2 Homework #2

Define a function \( \text{POLAR DUALITY} : P \rightarrow P^* \). Determine whether this function is a bijection for the following domains:

1. Set of all closed convex sets containing 0.
2. Set of all closed convex sets containing 0 in the interior.
3. Set of all polyhedra containing 0.
4. Set of all polyhedra containing 0 in the interior.
5. Set of all polytopes containing 0.
6. Set of all polytopes containing 0 in the interior.

3.3 Alternate Proof of Farkas’ Lemma

3.3.1 Homogeneous case

**Theorem 3.3.** \( a_1^T x \geq 0, a_2^T x \geq 0, \ldots, a_m^T x \geq 0 \implies C^T x \geq 0 \) iff \( C = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \forall i : \lambda_i \geq 0 \).

We first present a few useful definitions.

**Definition 3.2.** A cone is a convex set such that, \( \forall x \in \text{Cone}, \lambda x \in \text{Cone} \forall \lambda \geq 0 \). Alternately, a cone is an intersection of half-spaces defined by hyperplanes passing through the origin.

**Definition 3.3.** A polyhedral cone is a cone defined by a finite number of hyperplanes.

A cone may be finitely generated by points \( x_1, x_2, \ldots, x_k \) as follows:

\[
\text{CONE}(x_1, x_2, \ldots, x_k) = \{ y \mid y = \sum_{i=1}^{k} \lambda_i x_i, \forall i : \lambda_i \geq 0 \}
\]

**Proof of Farkas’ Lemma.** The first direction is easy:

If \( C = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \forall i : \lambda_i \geq 0 \), then \( C^T x = \sum_i \lambda_i a_i^T x \geq 0 \).

For the other direction, we show that if \( C \neq \lambda_1 a_1 + \cdots + \lambda_m a_m \), then \( C^T x \leq 0 \). Equivalently, if \( C \notin \text{CONE}(a_1, a_2, \ldots, a_m) \) then \( C^T x < 0 \).

Let \( d\delta \) be a vector such that:

1. \( d^T z > \delta \forall z \in \text{CONE}(a_1, a_2, \ldots, a_m) \)
2. \( d^T C < \delta \)
Since $0 \in \text{CONE}(a_1, a_2, \ldots, a_m)$, $d^T 0 > \delta$. WLOG, let $\delta = -1$.

Now we have that $d^T z > -1$ and $d^T C < -1$.

We know, therefore, that $d^T a_1 > -1$. Therefore, $\frac{1}{\tau} a_1 \in \text{CONE}(a_1, a_2, \ldots, a_m)$, so $d^T (\frac{1}{\tau} a_1) > -1$. Multiplying both sides by $\epsilon$, we have: $d^T a_1 > \epsilon$.

In the limit, $d^T a_1 \geq 0$, since this inequality is true for all epsilon. Therefore, we have a $d^T$ such that $d^T a_1 \geq 0, d^T a_2 \geq 0, \ldots, d^T a_m \geq 0$ but $C^T d < 0$. \hfill \Box

### 3.3.2 Non-homogeneous case

**Theorem 3.4.** $a_1^T x \geq b_1, a_2^T x \geq b_2, \ldots, a_m^T x \geq b_m$ $\implies$ $C^T x \geq d$ if $C = \sum_{i=1}^{m} \lambda_i a_i, \lambda_i \geq 0$ and $d \leq \sum_{i=1}^{m} \lambda_i b_i$

**Remark.** As the notetaker, I could not follow this argument. My notes reflect this, and my write-up reflects my notes. Therefore, I recommend looking at Schrijver’s notes on combinatorial optimization, which contain an alternate proof of this theorem.

**Proof.** As a helpful step, we show that for any $z \geq 0, a_1^T x - b_1 z \geq 0, \ldots, a_m^T x - b_m z \geq 0$ $\implies$ $C^T x \geq d z$. If we can show this, then we simply apply the homogeneous version and we’re done.

Case 1: $z > 0$

Case 2: $z = 0$, so $a_1^T x \geq 0 \ldots a_m^T x \geq 0$

$x_1 + \lambda x, \lambda \geq 0$

$C^T (x_1 + \lambda x) \geq d$, so $C^T x \geq 0$.

Consider the $m + 1$ dimension vector $(C, -d) = \lambda_1 (a_1, -b_1) + \lambda_2 (a_2, -b_2) + \cdots + \lambda_m (a_m, -b_m) + \lambda_{m+1} (0, 1)$

$C = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$

$-d = -b_1 \lambda_1 - b_2 \lambda_2 + \cdots + -b_m \lambda_m + \lambda_{m+1}$

$-d \geq -(b_1 \lambda_1 + b_2 \lambda_2 + \cdots + b_m \lambda_m)$

$d \leq b_1 \lambda_1 + b_2 \lambda_2 + \cdots + b_m \lambda_m$ \hfill \Box

### 3.4 Applications of Farkas’ Lemma

**Remark.** As a notetaker, I didn’t understand all of this, either.

Consider a business owned by $N = \{1, 2, \ldots, n\}$ partners. The profit they make is $P(N)$. Any subset of them $S \subset N$ working together could make a profit of $P(S)$. Therefore, in dividing up the profits, each subset $S$ must receive at least $P(S)$ or they would have incentive to go off and start their own business.

In other words, a solution to this profit-dividing problem (if one exists) must meet the following criteria: $P(N) = P_1 + P_2 + \cdots + P_n$ such that $\forall S, \sum_{i \in S} P_i \geq P(S)$. $P_i$ in this case is the amount of profit that goes to the $i$th partner.
**Definition 3.4.** The core of this game is a set of all solutions such that $P(S) \geq 0$ and $P(T) \geq P(S)$ for any $T \supseteq S$.

**Definition 3.5.** In a balanced game, there exists a fractional decomposition $N = \lambda_1 S_1 + \lambda_2 S_2 + \cdots + \lambda_k S_k$, where $\forall j \sum_{i:j \in S_i} \lambda_i = 1$ and $P(N) \geq \sum_{i=1}^{k} \lambda_i P(S_i)$.

**Theorem 3.5 (Bondareva-Shapley).** The core is non-empty if and only if the game is balanced.

**Proof.** First, we show that a core implies a balanced game. $P(N) = P_1 + P_2 + \cdots + P_n$.

$$\sum_{i=1}^{k} \lambda_i P(S_i) \leq \sum_{i=1}^{k} \lambda_i \sum_{j \in S_i} P_j$$

We can reorder the sums to obtain:

$$\sum_{j} P_j \sum_{i:j \in S_i} \lambda_i = \sum_{j} P_j = P(N)$$

In the reverse direction, we show that an empty core implies an imbalanced game.

$$-(P_1 + P_2 + \cdots + P_n) \geq -P(N) \rightarrow \lambda$$

$$\forall S \sum_{i \in S} P_i \geq P(S) \rightarrow \lambda_S$$

$$\forall i - \lambda + \sum_{S:i \in S} \lambda_S = 0 ; \lambda, \lambda_S \geq 0$$

$$-\lambda P(N) + \sum \lambda_i P(S) > 0 ; \lambda > 0$$

Then just apply Farkas’ lemma. □

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