Lecture 1

CS522: Advanced Algorithms

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1.1 Plan for the week

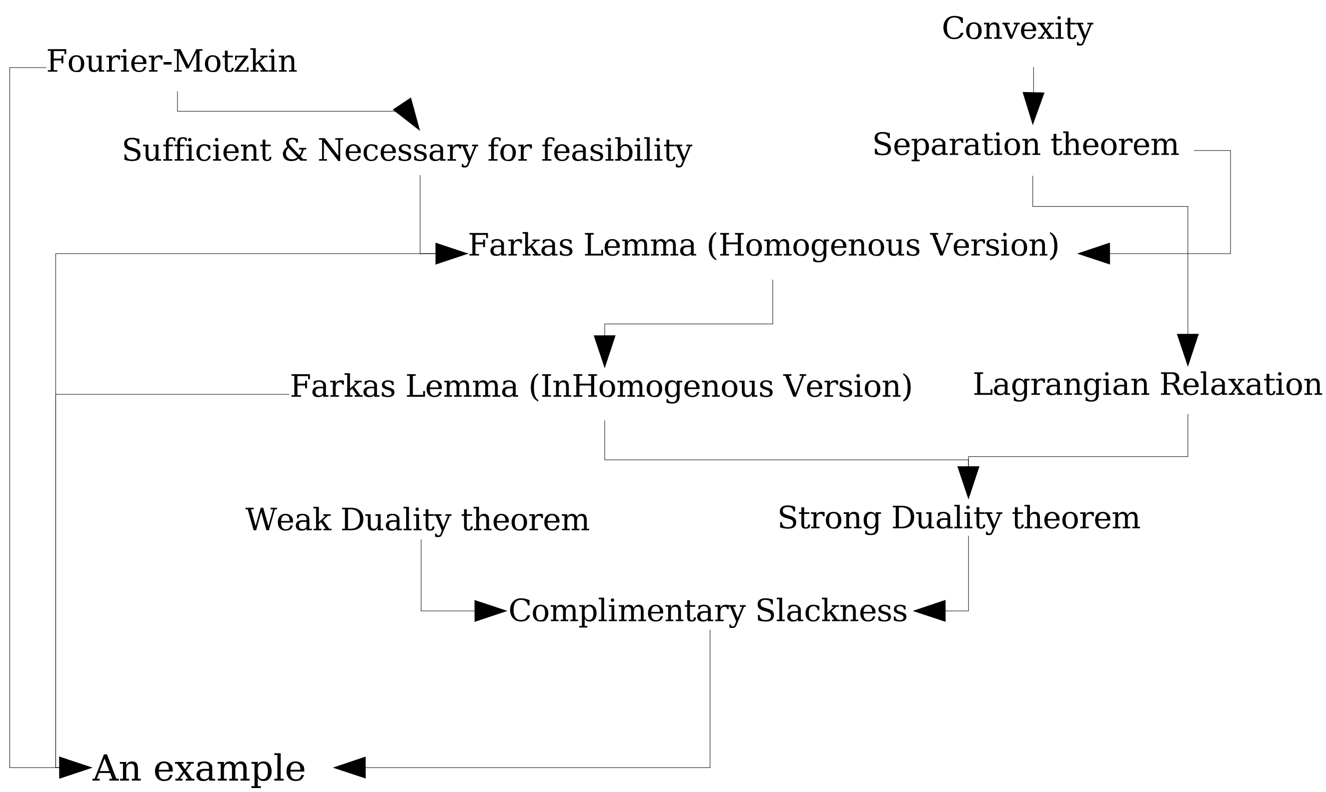

Figure 1.1: Plan for the week

The underlined tools, weak duality theorem and complimentary slackness, are most frequently used in CS. Strong duality is not often used in CS.

Remark. A nice reference for this material are Schrijver's Course notes [**?**]. The title is *A Course in Combinatorial Optimization*. It is available on the web.

1.2 Convex Sets

Definition 1.1. A Set S $\subset$ of $\mathbb{R}^n$ is *convex* if and only if $\forall x, y \in S$ the line connecting them is also in S. That is $\forall \lambda \in [0,1]$ $\lambda x + (1-\lambda)y \in S$.

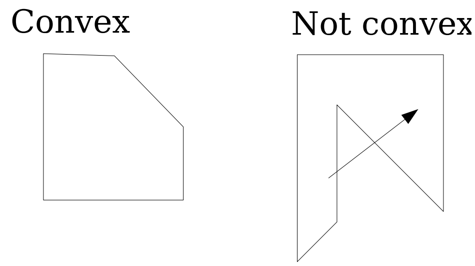

Figure 1.2: Examples of Convex and Non-Convex Spaces

In this course we will be assuming that all such convex sets are closed with respsect to the usual topology on $\mathbb{R}^n$.

Convex sets have nice properties. One such property for linear functions on convex sets is that every local max (min) is also a true max (min). This property does not hold in the non-convex case.

The first exercise of the lecture is to show the seperation thereom. This theorem states that given a convex set X and a point z not inside this convex set, there is a hyperplane such that z is on one side and X is on the other. Intuitively and pictorally this theorem is obvious, but making it formal will require some work.

Theorem 1.1 (Separation Theorem). *Let X convex $\subset R^n$ and $z \notin X$. There exist weights a_i such that $\forall x \in X$ $\sum_i^n a_i x_i \leq \delta$ and $\sum_i^n a_i z_i \geq \delta$ hold.*

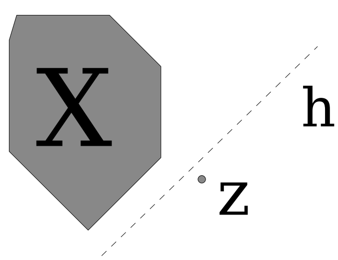

Figure 1.3: Examples of Separation

Proof. Take $x \in S$ such that $||x - z||$ is minimized. Notice such an x exists because the set is closed and therefore compact and we are minimizing a continuous function. We claim it is unique. For contradiction assume that there are x_1, x_2 such that $|z - x_1| = |z - x_2| = min$. Now consider their average, $y = \frac{x_1+x_2}{2}$. It is in the set X, by convexity and the fact that it lies on the line between x_1, x_2. This induces a triangle which is bisected by the line from y to z. It is clear that this vector is strictly shorter than the other two. This is a contradiction to their assumed minimality.

Now we have a unique minimal point x. Now draw a ball with radius $\delta = |x - z|$. Consider the tangent to this ball at the point x. We claim that this tangent hyperplane touches the convex set only at this one point. Moreover all of X is on one side of it. Suppose not, let y be the point that intersects the tangent hyperplane. Now draw the line xy. Notice that it must intersect the ball we have drawn, by convexity this spot on the ball is in the convex set, giving us a second minimal distance point, a contradiction to the above. Now we simply translate the hyperplane $\frac{\delta}{2}$ in the $z - x$ direction, this is our separating hyperplane. □

Definition 1.2. A *Half space* is given by the equation $\sum_j C_j x_j \leq \delta$.

Any convex set S can be written as the intersection over some (possibly infinite - even uncountably many) family of half-spaces. Notice the intersection of these sets is clearly convex since each halfspace is convex and therefore the line connecting any two points in their intersection is as well. Thus any collection of halfspaces is a convex set. Also, the above seperation theorem implies that we can continue using seperating hyperplanes (and then the inequality) to define the convex set, simply evaluate over all points not in the set.

Definition 1.3. A *Polyhedron* is a convex set which is the intersection of finitely many half spaces.

Remark. What we have above is a CoNP certificate of non-membership in the set of convex spaces. What we now want is an NP certificate for membership.

Definition 1.4. Convex Hull of a Set S, which may or not may be convex, will be denoted conv(S). conv(S) = $\{y | y = \sum^k \lambda_i x_i$ and $\forall i \; \lambda_i \geq 0$ and $\sum_i \lambda_i = 1\}$. k is an abitrarily large constant.

Remark. An alternative definition is that conv(S) = $\bigcap$S' is convex and $_{S \subset S'} S'$.

Our question is, how do we produce a certficate that a given element z is in a convex set S.

Theorem 1.2 (Carathéodory). *For any $z \in conv(S)$. z can be written as $z = \sum_i^k \lambda_i x_i$. $k \leq n + 1$ where n is the dimension of the space.*

Proof. let $z = \sum_i^k \lambda_i x_i$, when $k > n + 1$. Write the series of vectors in $\mathbb{R}^{n+1}$, $x_i' = (1, x_i)$. Notice by hypothesis there are more than n+1 of them, this implies these new vectors are linearly dependent. Therefore we can write $\sum \mu_i x_i' = 0$. We notice that $\sum \mu_i = 0$ because the first term is the 1 in each column. This also implies that there exists atleast one $\mu_i > 0$ and $\mu_j < 0$ since they are not all zero. We use this difference to cancel out one term. This decreases the number of vectors, decreasing k and allowing the induction hypothesis to be used. □

Definition 1.5. If S is finite, we call the hull of S a *polytope*

Our goal is the following theorem:

Theorem 1.3 (Mikowsky-Weyl). *a polytope is a bounded polyhedron*

Definition 1.6. Let $v \in S$ where S is a convex set. v is a corner point if and only if it can not be written as a combination of two other points.

Definition 1.7. Let A_z be the matrix of constraints which are satisfied as equality.

Theorem 1.4. *z is a corner pointer if and only if $rank(A_Z) = n$*

Proof. Suppose z is a corner point but $rank(A_z) < n$. We will write z as a combination of two other points to show a contradiction. Since A is not of full rank, it has some non-trivial kernel. So we may write $A_z c = 0$ with $c \neq 0$. We have a little bit of slack on the inequalities and there are finitely many, so we can find a δ so that translation by that amount will not violate the constraints. Now we write c as the average of $z + \delta c$ and $z - \delta c$. By our definition of δ and c, we know that this satisfies the constraints. This is a contradiction to z being a corner point.

For the other direction, assume $rank(A_z) = n$ but z = $\frac{x+y}{2}$ with x and y in the set. We must be able to write it this way because it is not a corner point. We will show that A_z is not injective, a contradiction to it being of full rank. $\forall row_i \in A_z$ $a_i * x \leq b_i = a_i * z$ and $a_i * y \leq b_i * z$ which implies $a_i * (x - z) \leq 0$ and $a_i * (y - z) \leq 0$ but, we no from above that $x + y = 2z$ so, $x - z = -(y - z)$ that is $A_z(x - z) = A_z(y - z) = 0$ By linearity, $A_z(x) = A_z(y) = A_z(z)$, a contradiction to the assumed rank. $\square$

We desire the implication: Bounded polyhedron implies polytope. We show something stronger from Schrijver's notes [**?**].

Lemma 1.5. *P a polyhedron with vertexes $p_1, ..., p_t$ implies $P = conv(p_1, ..., p_t)$.*

Remark. As we saw in class, the main intuition is that we go to the boundaries of the polyhedron and intersect in two points, x and y in the proof. We then use our induction hypothesis to write these as elements of the hull. We can do this since the sides since they are of strictly smaller dimension equivalently more constraints are satisfied. This is the condition $rank(A_z) > rank(A_x)$.

Proof. Since P is convex and each of the vertexes belong to P, $conv(p_1, ..., p_t) \subseteq P$. So we must show that given $z \in P$ then $z \in conv(p_1, ..., p_t)$ We are going to use induction on $n - rank(A_z)$, n the number of dimensions.

Base Case $n - rank(A_z) = 0$, this implies from the above that z is a corner point. So it is one of the p_i and is in the hull.

Inductive Case $n - rank(A_z) > 0$, this implies there is an element in the kernel of A_z call it c.

Denote by $\mu_0 := max\{\mu | z + \mu c \in P\}$ and $\eta_0 := max\{\eta | z + \eta c \in P\}$. This number exist by compactness of P. Let $x = z + \mu_0 c$ and $y = z - \eta_0 c$.

Notice now $\mu_0 = min\{\frac{b_i - a_i z}{a_i c} | a_i$ is a row of $A_i; a_i c > 0\}$. This is because μ_0 is the largest μ such that $a_i(z + \mu c) \leq b_i$ for each $i = 1, ..., m$. Written another way, $\mu \leq \frac{b_i - a_i z}{a_i c}$ for every i.

Let the minimium be attained by i_0. So for i_0 we have equality in the above minimum. This implies the following two facts:

$A_z x = A_z z + \mu_0 A_z c = A_z z$ and $a_{i_0} x = a_{i_0}(z + \mu_0 c) = b_{i_0}$. So A_x contains all rows in A_z and in particular a_{i_0}. $A_z c = 0$ while $a_{i_0} c \neq 0$. This implies $rank(A_x) > rank(A_z)$, So by our induction hypothesis x is in $conv(x_1, ..., x_t)$. Similarily, y belongs to $conv(x_1, ..., x_t)$, therefore z belongs to $conv(x_1, ..., x_t)$. $\square$

Theorem 1.6. *A polytope is a bounded polyhedron* *sketch.* Let $P = conv(x_1, \ldots, x_k)$ be a polytope. Induction takes care of the case when P is not the full dimension of the space. This implies there is a ball B with radius r centered inside P, (for ease assume centered at 0). Now define the set $P^* = \{y | x^T y \leq 1 \forall x \in P\}$. We claim P^* is a polyhedron. Write $P^* = \{y | x_i^T y \leq 1 \forall i\}$. We can write $x = \sum_i \lambda x_i$ because x is in the hull. Now this is a polyhedron and y is in it because $x^T y = \sum_i \lambda_i x_i^T y \leq \sum \lambda_i = 1$. The last inequality follows from the definition of P^* above. Now we notice that P^* is in fact bounded, because for any y (not equal to 0) in P^* we can scale it inside the ball. Thus we have there is some $x = y \frac{r}{||y||}$ so $x^T y \leq 1$ this implies that all of P^* is contained in the ball $B(0, \frac{1}{r})$. This means P^* is a bounded polyhedron implies it is a polytope by previous thereom. We will know show that $P^{**} = P$ and this will complete the theorem.

□

1.3 Homework

Given an ordered list of points $p_1, ..., p_k \in \mathbb{R}^2$. You may assume that P_1 is the last element of the list. Give algorithms for the following:

Question 1 Determine whether the boundary is not self-intersecting.

Question 2 Given a point x, is x contained in the polygon P defined by the above points.