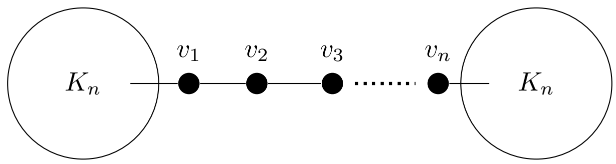
CSE 521: Design and Analysis of Algorithms **Fall 2023**

Problem Set 7

Deadline: Nov 27th in Canvas

1) a) Let G be a graph with $3n$ vertices that is a union of two disjoint copies the complete graph with n vertices, K_n, connected by a path of length n. Show that $\lambda_2(\tilde{L}_G) \leq O(1/n^3)$, where $\tilde{L}$ is the normalized Laplacian matrix.



b) For $d = 3, \ldots, 10$ and $n = 10000$ construct a random d-regular graph by taking the union of d random perfect matching. Output the second largest eigenvalue of the normalized adjacency matrix, i.e., A/d of each graph. Then, implement the power method on the PSD matrix $(A/d)^2$ to approximate 2nd largest eigenvalue of each matrix. Observe that the largest eigenvector of A/d is the all-ones vector. So, the second largest eigenvalue of A/d (in absolute value) is the square root of the second largest eigenvalue of $(A/d)^2$. Print your code, true and approximate eigenvalues.

2) We say a graph G is an expander graph if the second eigenvalue of the normalized Laplacian matrix ($\tilde{L}_G$), λ_2 is at least a constant independent of the size of G. It follows by Cheeger's inequality that if G is an expander, then $\phi(G) \geq \Omega(1)$ independent of the size of G. It turns out that many optimization problems are "easier" on expander graphs. In this problem we see that the maximum cut problem is easy in *strong* expander graphs. First, we explain the expander mixing lemma which asserts that expander graphs are very similar to complete graphs.

Theorem 7.1 (Expander Mixing Lemma)**.** *Let G be a d-regular graph and $1 = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq -1$ be the eigenvalues of the normalized adjacency matrix of G, A/d. Let $\lambda^* = \max\{\lambda_2, |\lambda_n|\}$. Then, for any two disjoint sets $S, T \subseteq V$,*

$$\left| |E(S,T)| - \frac{d \cdot |S| \cdot |T|}{n} \right| \leq d \cdot \lambda^* \sqrt{|S||T|}.$$

Note that $d|S||T|/n$ is the expected number of edges between S, T in a random graph where is an edge between each pair of vertices i, j with probability d/n. So, the above lemma says that in an expander graph, for any large enough sets $|S|, |T|$, then the number of edges between S, T is very close to what you see in a random graph.

Use the above theorem to design an algorithm for the maximum cut problem that for any d regular graph returns a set T such that

$$|E(T, \overline{T})| \geq (1 - 4\lambda^*) \max_S |E(S, \overline{S})|.$$

Note that the performance of such an algorithm may be terrible if $\lambda^* > 1/4$, but in strong expander graphs, we have $\lambda^* \ll 1$; for example, in Ramanujan graphs we have $\lambda^* \leq 2/\sqrt{d}$. So the number of edges cut by the algorithm is very close to optimal solution as $d \to \infty$. It turns out that in random graph $\lambda^* \leq 2/\sqrt{d}$ with high probability. So, it is easy to give a $1 + O(1/\sqrt{d})$ approximation algorithm for max cut in most graphs.