CSE 521: Design and Analysis of Algorithms **Fall 2023**

Problem Set 6 (Midterm)

Deadline: Oct 14th in gradescope

a) Consider the unit radius n-dimensional sphere S_n. Prove that it has a 1/4-net of size $2^{O(n)}$, i.e., there exists a set $N \subset S_n$ such that $|N| \leq 2^{O(n)}$ and for each point $x \in S_n$, there exists a point $y \in N$ such that $\|x - y\|_2 < 1/4$.

Hint: Use a greedy algorithm to construct N. Feel free to use that the volume of an n-dimensional ball of radius r is $c_n \cdot r^n$ where c_n is a constant independent of r.

b) Let $M \in \mathbb{R}^{n \times n}$ be a **symmetric** matrix and let N be a 1/4-net of the unit n-dimensional sphere. Show that

$$\sigma_1(M) \leq 2 \max_{y \in N} |y^T M y|,$$

where $\sigma_1(M)$ is the largest singular value of M; it is also the largest eigenvalue of M in absolute value.

c) Let $r_1, \ldots, r_n$ be n independent Radamacher random variables, and let $a_1, \ldots, a_n \in \mathbb{R}$. Show that for any $t \geq 0$,

$$\mathbb{P}\left[\left|\sum r_i a_i\right| > t\right] \leq 2e^{-\frac{t^2}{2\sum a_i^2}}.$$

d) Let A be the adjacency matrix of $G(n, 1/2)$, i.e., a random graph where there is an edge between every pair of vertices, independently, with probability 1/2. Use the previous parts to show that, for some universal constant C (independent of n),

$$\mathbb{P}\left[\|A - \mathbb{E}[A]\| \leq C\sqrt{n}\right] \geq 1 - 2^{-\Omega(n)}.$$

Hint: Note that proving the above claim may seem impossible because you have to prove a concentration over **infinitely** many vectors. However, if you use a net as suggested in previous parts, you can reduce the question to finitely many vectors.