Algorithms:

simplex } "smoothed" poly time

ellipsoid

interior pt methods } polynomial time

smoothed analysis of algs:
measures perf of algorithm under slight random perturbations of input!

Using LP to design approx algs

Example 1: Weighted vertex cover - deterministic rounding

$\min \sum_v w_v y_v$

$y_u + y_v \geq 1 \quad \forall (u,v) \in E$

$y_u \in \{0,1\} \quad \forall u \in V$ $\xrightarrow[\text{relax to LP}]{}$

$\min \sum_v w_v x_v$

$x_u + x_v \geq 1 \quad \forall (u,v) \in E$

$0 \leq x_u \quad \forall u$

OPT int program $\geq$ OPT relaxation

Relax, solve, round & bound

x^* opt for relaxation

set $y_v = \begin{cases} 1 & x_v^* \geq \frac{1}{2} \\ 0 & \text{o.w} \end{cases}$

Claim: $\vec{y}$ feasible since $\forall$ edge (u,v) at least one of $x_u^*, x_v^* \geq \frac{1}{2}$

$\sum_v w_v y_v \leq 2 \sum_v w_v x_v^* \leq 2\, OPT_{relaxation} \leq 2\, OPT_{IP}.$

.

Integrality gap of LP relaxation: $\max_{\text{instances}} \frac{\text{OPT integer soln}}{\text{OPT fractional sol'n}}$

This relaxation has $IG = 2 - \frac{2}{n}$ clique w/ $w_v = 1\ \forall v$

fractional soln $x_v = \frac{1}{2}\ \forall v$ $\frac{n}{2}$

integer soln $y_v = 1\ \forall v$ except 1 $n-1$

int gap $\geq \frac{n-1}{\frac{n}{2}}$

Example 2: Max 2-SAT - randomized rounding

max version NP-hard

n vars $x_1, \dots x_n$ max $\sum z_j$

m clauses $z_1, \dots z_n$

$y_{j_1} + y_{j_2} \geq z_j$

first literal, second literal

$x_i = \begin{cases} 1 & \text{var } i \text{ T} \\ 0 & \text{F} \end{cases}$ $z_j = \begin{cases} 1 & \text{if clause } j \text{ satisfied} \\ 0 & \text{o.w.} \end{cases}$

$y_{j_1} = \begin{cases} x_{j_1} & \text{if } + \\ 1 - x_{j_1} & \text{if negated} \end{cases}$

$0 \leq x_i \leq 1$

$0 \leq z_j \leq 1$

Simple alg:

treat x_j as probability

set var $j = \begin{cases} T & \text{with prob } x_j \\ F & \text{o.w.} \end{cases}$

Claim: $E(\#\text{ clauses satisfied}) \geq \frac{3}{4}\text{OPT}$

via $\Pr(j^{th} \text{ clause satisfied}) \geq \frac{3 z_j}{4}$

at LP opt $z_j = \min\{1, x_r + x_s\}$ $\equiv$ make z_j as big as possible

supposing j^{th} clause $(x_r \vee x_s)$

$\Pr(\text{clause satisfied}) = 1 - (1-x_r)(1-x_s) = x_r + x_s - x_r x_s$

$\geq x_r + x_s - \frac{1}{4}(x_r + x_s)^2 \geq z_j - \frac{1}{4} z_j^2 \geq \frac{3}{4} z_j$

AMGM $\frac{x_r + x_s}{2} \geq \sqrt{x_r x_s}$

$\frac{x+y}{2} - \sqrt{xy}$

$\frac{1}{2}(\sqrt{x} - \sqrt{y})^2 \geq 0$

$\Rightarrow \frac{(x_r + x_s)^2}{4} \geq x_r x_s$

Int gap $\frac{3}{4}$

$x_1 \vee x_2$, $x_1 \vee \bar{x}_2$, $\bar{x}_1 \vee x_2$, $\bar{x}_1 \vee \bar{x}_2$

Set $x_1 = x_2 = \frac{1}{2}$

$z_j = 1 \quad \forall j$

but only 3 clauses can be satisfied

Input: n pts, distance fn satisfying triangle inequality, cost f_i $\forall 1 \le i \le n$

$$\min \sum_i \sum_r y_i^{(r)} (r + f_i) \qquad\qquad \max \sum_j \alpha_j$$

$$\sum_i \sum_{r \mid d(i,j) \le r} y_i^{(r)} \ge 1 \quad \forall j$$

$$(*) \quad \sum_{j \mid d(i,j) \le r} \alpha_j \le r + f_i \quad \forall i, r$$

$$y_i^{(r)} \ge 0 \quad \forall i, r. \qquad\qquad \alpha_j \ge 0 \quad \forall j$$

$\boxed{\text{Primal-dual alg:}}$ [Charikar, Panigrahy]

Initially $\alpha_j = 0 \; \forall j$ — dual feasible; $y_i^{(r)} = 0 \; \forall i, r$ — primal infeasible

Increase all α_j's as long as $(*)$ satisfied. (arbitrarily)

if some $(*)$ constraint goes tight, set $y_i^{(r)} = 1$

and stop ↑ any α_j that contributes to that constraint.

when no longer possible to ↑ anything, primal & dual feasible

if some j not covered, then it doesn't participate in any tight constraint.
alg can't have terminated

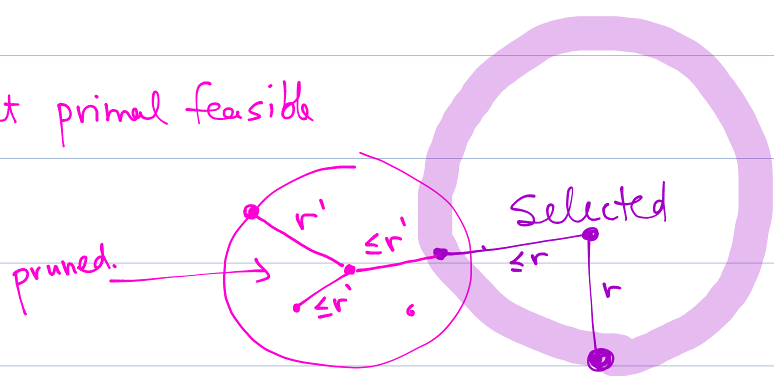
To finish things off, prune clusters:

Consider clusters constructed in ↓ order of radii (tie-breaking arbitrary)

If current cluster doesn't intersect any clusters previously selected, choose it,
i.e. ∃ pt in both
o.w. toss it.

Triple radius of all selected clusters.

Claim: result primal feasible



If cluster i with radius r selected (before radius tripled)

$$\sum_{j \mid j \in C_i(r)} \alpha_j = r + f_i$$

$j \notin C_k(r_k)$ for any other selected cluster

$$\Rightarrow \sum_{(i, r_i) \text{ selected}} (f_i + 3r_i) \le 3 \sum_j \alpha_j \le 3\,OPT$$