Algorithms:

- Simplex
- Ellipsoid
- Interior pt methods

"smoothed" poly-time

smoothed analysis

measured part of algorithm under slight random perturbation of input.

Using LP to design approx algo

**Example 1:** Weighted vertex cover - deterministic rounding

\[
\min \sum_v w_v y_v \\
\min \sum_v w_v x_v
\]

\[
y_v + x_v \geq 1 \quad \forall (v, e) \in E
\]

\[
x_v + x_e \geq 1 \quad \forall (e, e') \in E
\]

\[
y_v \in \{0, 1\} \quad \forall v \in V
\]

Opt int program \implies Opt relaxation

Relax, solve, round & bound

\[x^* \text{ opt for relaxation}
\]

\[
\text{set } y_v = \begin{cases} 1 & x_v^* \geq \frac{1}{2} \\ 0 & \text{otherwise}
\end{cases}
\]
Claim: feasible since 
\[ u \leq \frac{1}{2} \]

\[ \sum_{v} w_v y_v \leq 2 \sum_{v} w_v x_v \leq 2 \text{OPT}_{\text{relaxation}} \leq 2 \text{OPT}_{\text{IP}}. \]

Integrality gap of LP relaxation:
\[
\frac{\text{OPT integer soln}}{\text{OPT fractional soln}}
\]

This relaxation has \( IG^- = 2 - \frac{2}{n} \). Clear if \( w_v = 1 \) \( \forall v \)

fractional soln \( x_v = \frac{1}{2} \) \( \forall v \)

integer soln \( y_v = 1 \) \( \forall v \) except \( 1 \) \( n-1 \)

\[ m^+ \geq \frac{n-1}{n} \]

[Drafted by an assistant]
Example 2: Max 2-SAT - randomized rounding

Max version NP-hard

\[ \text{max } \sum z_j \]

\[ \text{n vars } x_1, \ldots, x_n \]

\[ \text{m clauses } \ell_1, \ldots, \ell_m \]

\[ x_i \in \{0, 1\} \text{ if clause } j \text{ satisfied} \]

\[ y_{i,j} \geq z_j \]

\[ 0 \leq x_i \leq 1 \]

\[ 0 \leq z_j \leq 1 \]

Simple alg:

1. treat \( x_j \) as probability
2. set \( j = \begin{cases} T \text{ with prob } x_j \\ F \text{ o.w.} \end{cases} \)

Claim: \( E(\# \text{ clauses satisfied}) \geq \frac{3}{4} \text{ OPT} \)

via \( \text{Pr}(j^{th} \text{ clause satisfied}) \geq \frac{3z_j}{4} \)

at \( \text{LP opt } z_j = \min \{1, x_r + x_s\} \) \equiv \text{ make } z_j \text{ as big as possible }

supposing \( j^{th} \text{ clause } (x_r \lor x_s) \)
\[ \Pr(\text{clause satisfied}) = 1 - (1-x_r)(1-x_s) = x_r + x_s - x_rx_s \]

\[ \geq x_r + x_s - \frac{1}{4}(x_r + x_s)^2 \quad \Rightarrow \quad z_j - \frac{1}{4} z_j^2 \geq \frac{3}{4} z_j \]

\[ \text{AM-GM} \quad \frac{x_r + x_s}{2} \geq \sqrt{x_r x_s} \quad \quad \quad \frac{x_r - x_s}{2} \geq 0 \]

\[ \Rightarrow \quad \frac{(x_r + x_s)^2}{4} \geq x_r x_s \]

Int. gap \( \frac{3}{4} \)

\[ x_1, x_2, x_1x_2, \overline{x}_1 \overline{x}_2, \overline{x}_1x_2, x_1x_2 \]

Set \( x_1 = x_2 = \frac{1}{2} \)

\[ z_j = 1 \quad \forall j \]

but only 3 clauses can be satisfied
Input: n pts, distance fn satisfying triangle inequality, cost f; \( 1 \leq i \leq n \)

\[
\min \sum_{i} \sum_{j \in \mathcal{H}(i)} y_{ij}(r + f_{i}) \\
\max \sum_{j} \alpha_{j}
\]

\[
\sum_{i} \sum_{j \in \mathcal{H}(i)} y_{ij} \geq 1 \quad \forall j
\]

\[
\sum_{j} \alpha_{j} \leq r + f_{i} \quad \forall i, r
\]

\[
y_{ij} \geq 0 \\
\forall i, j
\]

\[\alpha_{j} \geq 0 \quad \forall j\]

Primal-dual alg. [Charikar, Panigrahy]

Initially \( \alpha_{j} = 0 \quad \forall j \)

\( y_{ij} = 0 \quad \forall i, r \) dual-feasible

\( y_{ij}^{(r)} = 0 \quad \forall i, r \) primal infeasible

Increase all \( \alpha_{j} \)’s as long as \((\ast)\) satisfied. (arbitrarily)

if some \((\ast)\) constraint goes tight, set \( y_{ij}^{(r)} = 1 \)

and stop if any \( \alpha_{j} \) that contributes to that constraint.
when no longer possible to \( \uparrow \) anything, primal & dual feasible

if some \( j \) not covered, then it doesn’t participate in any tight constraint. cdg can’t have terminated

To finish things off, prune clusters:

Consider clusters constructed in \( \downarrow \) order of radii (tie-breaking arbitrary)

If current cluster doesn’t intersect any clusters previously selected, choose it, i.e. \( k \) in both

O.w. toss it.

Triple radius of all selected clusters.

Claim: result primal feasible
If cluster $i$ with radius $r$ selected (before radius tripled)

$$\sum_{j \in \mathcal{C}(r)} \alpha_j = r + f_i$$

$j \notin \mathcal{C}_k(r)$ for any other selected cluster

$$\Rightarrow \sum_{(i,r) \text{ selected}} (f_i + 3r_i) \leq 3 \sum_j \alpha_j \leq 3 \text{OPT}$$