Maximum load

L_i # items that hash to location i in table

balls in bins problem

$$\Pr(L_i \geq k) \leq \sum_{\substack{T \text{ s.t.} \\ |T|=k}} \Pr(\text{all elts in } T \text{ hash to } i) = \binom{n}{k}\frac{1}{n^k} \leq \frac{n^k}{k!}\frac{1}{n^k} = \frac{1}{k!}$$

$$k! \geq k^{k/2} \geq \left(\frac{8\log n}{\log\log n}\right)^{\frac{4\log n}{\log\log n}} \geq \left[(\log n)^{1/2}\right]^{\frac{4\log n}{\log\log n}} = 2^{2\log n} = n^2$$

$$\boxed{k = \frac{8\log n}{\log\log n}}$$

$$\Rightarrow \Pr\left(\exists i \; L_i \geq \frac{8\log n}{\log\log n}\right) \leq n \cdot \frac{1}{n^2} \leq \frac{1}{n}$$

$$E\left(\max_i L_i\right) = \Theta\left(\frac{\log n}{\log\log n}\right).$$

Power of 2 choices

standard balls in bins: each ball $\rightarrow$ random bin

Instead: throw balls one at time

for each ball, pick $\underline{\underline{2}}$ random bins

place ball in bin that has fewer balls

n balls $\xrightarrow[\text{2 choice process}]{}$ n bins $\qquad E(\text{max load}) = \Theta(\log\log n)$

Intuition:

ball has height k if when it's placed, there are $k-1$ other balls there

expect $\leq \frac{n}{2}$ bins with at least 2 balls

$$\Pr(\text{ball has height} \geq 3) \leq \frac{1}{2} \cdot \frac{1}{2}$$

expect $\leq \frac{n}{2^2}$ bins with at least 3 balls

$$\Pr(\text{ball has height} \geq 4) \leq \frac{1}{2^2} \cdot \frac{1}{2^2} = \frac{1}{2^{2^2}}$$

$\vdots$

expect $\leq \frac{n}{2^{2^{h-1}}}$ bins with at least h balls

$$\Pr(\text{ball has height} \geq h+1) \leq \frac{1}{2^{2^{h-1}}} \frac{1}{2^{2^{h-1}}} = \frac{1}{2^{2^h}}$$

expect $\leq \dfrac{n}{2^{2^h}}$ bins of height $\geq h+1$

$\dfrac{n}{2^{2^h}} < 1$

$n < 2^{2^h}$

$\log n < 2^h$

$\log\log n < h$