Maximum weight matching

Bipartite graphs very useful for modeling:
- jobs & machines
- employers, employees
- men women

Matching: set of edges with no common endpoints

Total weight 5 8 5

Often 3 weights on edges
and we want to find max weighted
matching
Ascending auction alg for integer weights
for max matching

Fix bid increment $\delta = \frac{1}{n+1}$

$p^3 = (p_1, \ldots, p_n)$ $p_j$: price of item $j$

Initially all $p_j = 0$ and matching empty
$M(i) = \emptyset \forall i$

As long as matching not perfect

one unmatched bidder $i$

bids on some $j$ in

$$D_i(p^3) = \{j \mid v_{ij} - p_j \geq v_{ik} - p_k \text{ and } v_{ij} > p_j\}$$

demand set.

and bids $p_j + \delta$ on $i$.

If $j$ unmatched, then $M(i) := j$

else, say $M(k) = i$

remove $(k, j)$ from matching
and add $(i, j)$ [i.e. $M(i) := j$]
Theorem

Suppose $v_{ij}$ integer. The "auction algorithm" terminates with a max lot matching and the final prices are almost "envy-free"

$$M(i)=j \Rightarrow v_{ij}-p_j \geq v_{ik}-p_k-\delta \quad \forall k \quad \text{(\star)}$$

Proof

- from moment item is matched, it stays matched forever
- until item is matched, $p_j=0$

$\Rightarrow$ if $i$ unmatched, $D_i(D)=\emptyset$

Every step till termination $\quad \Delta (\sum p_j) = \delta$

$$p_k \leq \max_{i,j} v_{ij} - \delta \quad \Rightarrow \text{alg terminates}$$

$\Rightarrow$ matching perfect.

(\star) $\checkmark$

Let $M^{*}$ be any other matching, $M$ final matching output

by (\star) $\quad \sum_i (v_{im(i)} - p_{m(i)}) \geq \sum_i (v_{im^{*}(i)} - p_{m^{*}(i)} - \delta)$

$\Rightarrow \sum v_{im(i)} \geq \sum v_{im^{*}(i)} - \delta n \delta = \sum v_{im^{*}(i)} - \frac{n}{n-1}$
since $w$ is any p.m. is int $\Rightarrow$ $M$ max wt matching

How many times main loop executed? $\leq \frac{n}{6} \cdot \max_{ij} v_{ij}$

$\max_{ij} v_{ij} = 1 \Rightarrow O(n^2)$ times, i.e., matching