Optimization

$\max/\min\ f(x_1, .., x_n)$

subject to constraints

Example: $G=(V,E)$ quadratic programming problem

$\min \sum_{i=1}^{n} x_i$ one var / vertex

subject to $x_i(1-x_i)=0 \quad \forall i \in V$

$(1-x_i)(1-x_j)=0 \quad \forall (i,j) \in E$

This is the vertex cover problem NP-hard.

# The Diet Problem

athlete wants to max protein consumption

subject to $\leq 5$ units of fat/day

$\leq \$6$/day

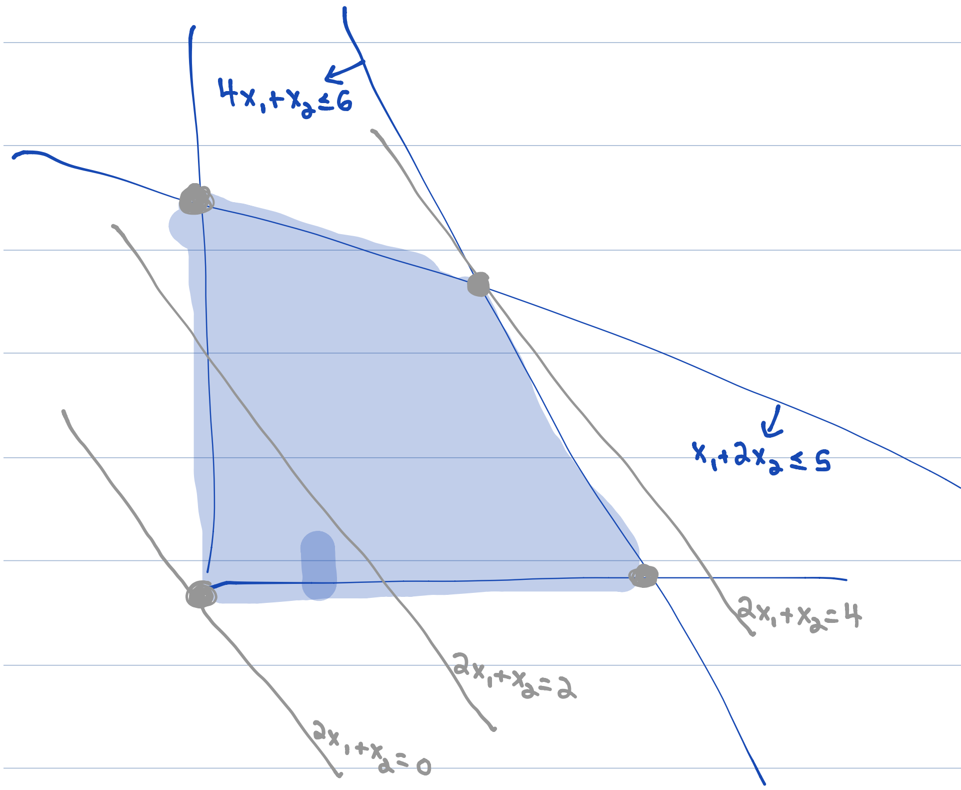
| | protein/lb | fat/lb | \$/lb |
|---|---|---|---|
| steak | 2 | 1 | 4 |
| peanut butter | 1 | 2 | 1 |

$x_1$ # lbs of steak/day

$x_2$ = # lbs of PB/day

$$\max\ 2x_1 + x_2$$

subject to
$$4x_1 + x_2 \leq 6$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$




max $\boxed{2x_1 + x_2}$ ← objective function

subject to

$$\boxed{\begin{array}{c} 4x_1 + x_2 \le 6 \\ x_1 + 2x_2 \le 5 \\ x_1, x_2 \ge 0 \end{array}}$$

← feasible set
feasible region

feasible pt with
maximum objective
fn value is "optimal solution"

polyhedron : intersection of half-spaces

if also bounded, nonempty $\Rightarrow$ polytope

feasible set is convex $\quad x, y \in S \Rightarrow \lambda x + (1-\lambda) y \in S$

$0 \le \lambda \le 1$

linear cost fns define family of parallel hyperplanes

optimal feasible pt must occur at corner a.k.a. vertex

(can't be expressed as convex combination of feasible pts)

Unfortunately,

can be too many vertices to enumerate $\approx m^{n/2}$

Example $\quad 0 \le x_i \le 1 \quad 1 \le i \le n$

n dimensional hypercube

Other examples:

1

Given $g_{ij}$ = grade of student $i$ in class $j$  $1 \le i \le n$, $1 \le j \le m$

Is there a "simple explanation"?

$a_i$ : aptitude of student $i$

$e_j$: easiness of class $j$

$$\sum_{i,j} |g_{ij} - a_i - e_j| \text{ small?}$$

$$s_{ij} = |g_{ij} - a_i - e_j|$$

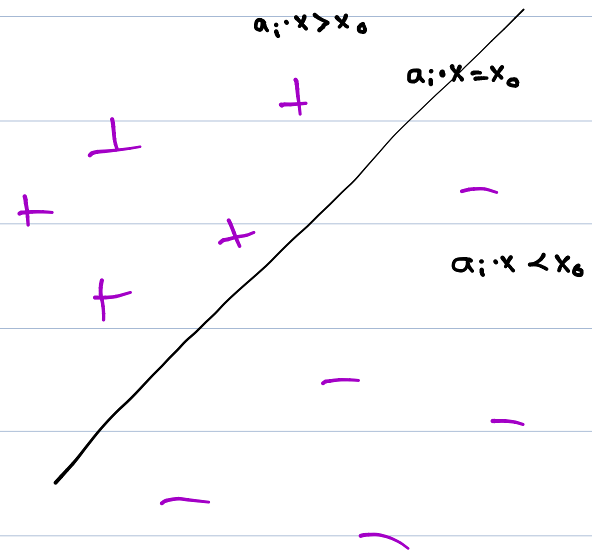
$$\min \sum s_{ij}$$

$$g_{ij} = a_i + e_j + \varepsilon_{ij}$$

$$-s_{ij} \le \varepsilon_{ij} \le s_{ij}$$

$$s_{ij} \ge 0$$

(2) m labeled examples $a_1, \ldots, a_m \in \mathbb{R}^n$




$l_1 \cdots l_m \qquad l_i \in \{-1, 1\}$

Is there a good linear classifier?

$$\text{sign}(a_i \cdot x - x_0) = l_i$$

$$l_i (a_i \cdot x - x_0) \geq 0$$

$a_i \cdot x - x_0 \geq 0 \quad \forall i \in \text{Positive}$

$a_i \cdot x - x_0 < -\omega \quad \forall i \in \text{Negative}$

Algorithms:

simplex } "smoothed" poly time

smoothed analysis of algs: measures perf of algorithm under slight random perturbations of input!

ellipsoid

interior pt methods

} polynomial time

Using LP to design approx algs

Example 1: Weighted vertex cover - deterministic rounding

## Example 2: Max 2-SAT - randomized rounding

max version NP-hard

n vars $x_1, \dots x_n$ max $\sum z_j$

m clauses $z_1, \dots z_n$ $y_{j_1} + y_{j_2} \geq z_j$

first literal, second literal

$x_i = \begin{cases} 1 & \text{var } i \text{ T} \\ 0 & \text{F} \end{cases}$ $z_j = \begin{cases} 1 & \text{if clause } j \text{ satisfied} \\ 0 & \text{o.w.} \end{cases}$

$y_{j_1} = \begin{cases} x_{j_1} & \text{if } + \\ 1 - x_{j_1} & \text{if negated} \end{cases}$

$0 \leq x_i \leq 1$

$0 \leq z_j \leq 1$

Simple alg:

treat $x_j$ as probability

set var $j = \begin{cases} T & \text{with prob } x_j \\ F & \text{o.w.} \end{cases}$

Claim: $E(\#\text{ clauses satisfied}) \geq \frac{3}{4}\,OPT$

via $Pr(j^{th} \text{ clause satisfied}) \geq \frac{3z_j}{4}$

at LPopt $z_j = \min\{1, x_r + x_s\}$ $\equiv$ make $z_j$ as big as possible

supposing $j^{th}$ clause $(x_r \vee x_s)$

$Pr(\text{clause satisfied}) = 1 - (1-x_r)(1-x_s) = x_r + x_s - x_r x_s$

$$\geq x_r + x_s - \frac{1}{4}(x_r + x_s)^2 \geq z_j - \frac{1}{4} z_j^2 \geq \frac{3}{4} z_j$$

AMGM $\frac{x_r + x_s}{2} \geq \sqrt{x_r x_s}$

$\Rightarrow \frac{(x_r + x_s)^2}{4} \geq x_r x_s$