$P = \{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq 0\}$ A $m \times n$ matrix; $b \in \mathbb{R}^m$

$x \in P$ extreme pt of P iff $\nexists\, y \in \mathbb{R}^n$, $y \neq 0$ s.t. $x+y,\ x-y \in P$

also called vertex

Lemma $\vec{x} \in P$ is extreme pt iff $\exists$ n linearly indep tight constraints at $\vec{x}$

Proof:

<u>Claim:</u> $x,\ x+y,\ x-y \in P$ $y \neq 0$ $\Rightarrow a_i \cdot y = 0$ $\forall$ tight constraint

<u>Pf:</u> $a_i \cdot (x+y) = b_i$
$a_i \cdot (x-y) = b_i$ $\Rightarrow a_i \cdot y = 0$

$\Rightarrow$ if $\exists$ n LI tight constraints $\Rightarrow \nexists\, y$ orthogonal to all $\Rightarrow$ extreme pt

if $\nexists$ n LI tight constraints $\Rightarrow \exists\, y$ orthogonal to all
$\Rightarrow$ for $\varepsilon > 0$ sufficiently small $x + \varepsilon y,\ x - \varepsilon y \in P$

Corollary: If $\vec{x}$ is extreme pt soln where all $x_i > 0$

then |maximal # of LI. tight constraints| = # vars

Application: Max weighted matching

first integer program

$$\max \sum_i \sum_j v_{ij} y_{ij}$$
$$\sum_i y_{ij} \le 1 \quad \forall j$$
$$\sum_j y_{ij} \le 1 \quad \forall i$$
$$y_{ij} \in \{0,1\}$$

$\xrightarrow{\text{relax to LP}}$

$$\max \sum_i \sum_j v_{ij} x_{ij}$$
$$\sum_i x_{ij} \le 1 \quad \forall j$$
$$\sum_j x_{ij} \le 1 \quad \forall i$$
$$x_{ij} \ge 0 \quad \forall (i,j)$$

Claim: LP is integral

while edges remain

Solve LP $\rightarrow$ extreme pt soln

in both of these cases stay optimal $\Big\{$

if $\exists\ x_{ij} = 0$, remove edge (i,j) from graph

if $\exists x_{ij} = 1$ add (i,j) to matching, remove i & j from vertex set

Suppose at some pt no integer var, m edges remain

every tight constraint of type $\sum x_{ij} \le 1$ includes ≥ 2 nonzero vars

$\Rightarrow$ # vars $\ge 2\max(n_1, n_2)$ $\Rightarrow$ at least this many LI tight constraints

herein $\rightarrow \leftarrow$

$\boxed{\Xi}$

Thm: If $\{\min cx \mid x \in P\}$ has optimal soln

$\exists$ extreme pt optimal soln

Pf of thm: Suppose no vertex opt soln

Consider opt soln x w/ maximum # of tight constraints

$\Rightarrow \exists y$ s.t. $x \pm y$ feasible $y \neq 0$

$c \cdot y = 0$ why?

$a_i \cdot y = 0$ $\forall$ tight constraints

wlog y has some j with $y_j < 0$ (o.w. take $-y$)

($\Rightarrow x_j > 0$)

$\Rightarrow \exists$ largest $\alpha > 0$ s.t. $x + \alpha y$ is feasible

$c \cdot (x + \alpha y) = c \cdot x$

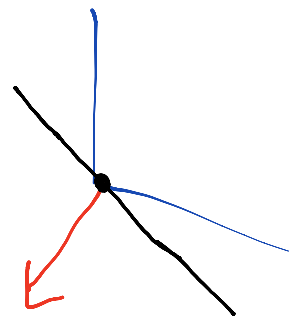
$a_i(x + \alpha y) = b_i$

Some new constraint becomes tight $\rightarrow\leftarrow$

Equivalent defn:

$\vec{x}$ is vertex of P if $x \in P$ and $\exists\, \vec{c}$ s.t.

$$c \cdot x < c \cdot y \qquad \forall\, y \in P \text{ s.t. } y \neq x$$



Algorithms return vertex optimal solns

Duality

$$\max \; 2x_1 + x_2$$

$$(1) \quad 4x_1 + x_2 \le 6$$

$$(2) \quad x_1 + 2x_2 \le 5$$

$$x_1, x_2 \ge 0$$

$(1,1)$ feasible

$\Rightarrow \; OPT \ge 3$

what about upper bounds?

$$(1) \Rightarrow OPT \le 6$$

$$2 \cdot (2) \Rightarrow 2(x_1 + 2x_2) \le 10 \qquad OPT \le 10$$

$$\frac{1}{2}(1) + \frac{1}{4}(2) \Rightarrow \underbrace{\frac{1}{2}(4x_1 + x_2) + \frac{1}{4}(x_1 + 2x_2)} \le \frac{6}{2} + \frac{5}{4}$$

$$2x_1 + x_2 \le \frac{5}{2}x_1 + x_2$$

$$\Rightarrow OPT \le 4\frac{1}{4}$$

What is the best upper bound we can get this way?

$$y_1(4x_1+x_2) \le 6y_1$$

$$y_2(x_1+2x_2) \le 5y_2$$

if $y_1(4x_1+x_2) + y_2(x_1+2x_2) \geq 2x_1+x_2$

$$\Rightarrow 6y_1+5y_2 \geq \text{OPT}$$

What is best upper bound we can get this way?

$$\left.\begin{aligned} &\min\ 6y_1+5y_2 \\ \text{subject to}\quad &4y_1+y_2 \geq 2 \\ &y_1+2y_2 \geq 1 \\ &y_1,y_2 \geq 0 \end{aligned}\right\}\ \text{dual LP}$$

(P)

$$\min\ c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1) \cdot y_1$$
$$(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2) \cdot y_2$$
$$\vdots$$
$$(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m) \cdot y_m$$

$$x_1, x_2, \ldots x_n \geq 0$$

(D)

$$\max\ b_1y_1 + b_2y_2 + \ldots + b_my_m$$

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \leq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \leq c_2$$
$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \leq c_n$$

$$y_1, y_2, \ldots, y_m \geq 0$$

$$\min\ c \cdot x$$
$$Ax \geq b$$
$$x \geq 0$$

$$\max\ b \cdot y$$
$$y^T A \leq c^T$$
$$y \geq 0$$

By construction, $OPT(D) \leq OPT(P)$ called weak duality

Weak Duality

x feasible for (P), y feasible for (D)

$$\Rightarrow b \cdot y \leq c \cdot x$$

Proof: $y^T b \leq y^T A x \leq c^T x$

Corollaries

x feasible for (P), y feasible for (D) $\quad b \cdot y = c \cdot x$

$\Rightarrow$ x opt for (P) $\quad y$ opt for (D)

dual unbounded $\Rightarrow$ primal infeasible

primal unbounded $\Rightarrow$ dual infeasible

Duality Thm

(P) & (D) primal-dual pair of LPs, then one of following holds

1. both infeasible
2. (P) unbounded, (D) infeasible
3. (D) unbounded, (P) infeasible
4. both feasible $\exists$ opt solns x^*, y^* s.t. $c \cdot x^* = b \cdot y^*$

Complementary Slackness

The following are equivalent:

(1) x^* opt for (P) & y^* OPT for (D)

(2) $\forall i \quad x_i^* \left(c_i - \sum_j y_j^* a_{ji}\right) = 0$

$\forall j \quad y_j^* \left(\sum_i a_{ji} x_i^* - b_j\right) = 0$

$$\max\ c \cdot x$$
$$Ax \le b$$
$$x \ge 0$$

primal (P)

$$\min\ b \cdot y$$
$$y^T A \ge c^T$$
$$y \ge 0$$

dual (D)

Duality

(P) and (D) feasible

Then $OPT(P) = OPT(D)$

Complementary Slackness

The following are equivalent

1) x^* opt for (P), y^* opt for (D)

2) $x_j^* \left(\sum_i y_i^* a_{ij} - c_j \right) = 0 \quad \forall j$

and

$y_i^* \left(b_i - \sum_j a_{ij} x_j^* \right) = 0 \quad \forall i$

$$\begin{aligned} & b \cdot y^* - c \cdot x^* \\ \ge\ & \sum_i \sum_j a_{ij} x_j^* \, y_i^* - \sum_j c_j x_j^* \\ =\ & \sum_j x_j^* \left(\sum_i a_{ij} y_i^* - c_j \right) \end{aligned}$$

$$\begin{aligned} & b \cdot y^* - c \cdot x^* \\ \ge\ & \sum_i b_i y_i^* - \sum_j x_j^* \sum_i y_i^* a_{ij} \\ =\ & \sum_i y_i^* \left(b_i - \sum_j a_{ij} x_j^* \right) \end{aligned}$$

Pf of duality thm:

We will use separating hyperplane thm:

Separating Hyperplane Thm

K closed convex set

$z \notin K$

Then $\exists$ hyperplane $d \cdot x = d_0$

separating z from K

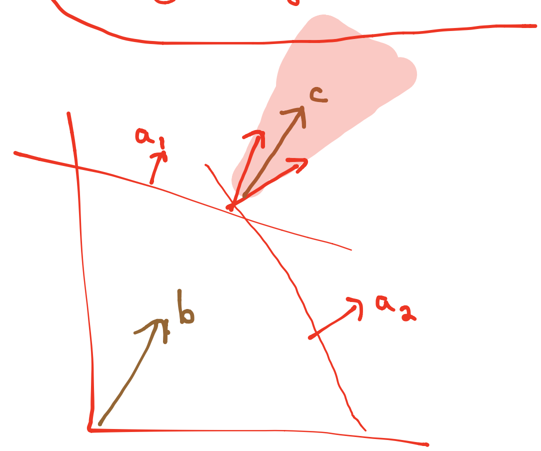
i.e. $d \cdot z > d_0$

$d \cdot x \leq d_0 \quad \forall x \in K$

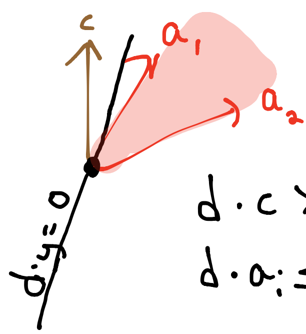
Suppose x^* optimal for (D)

$$I = \{ i \mid a_i x^* = b_i \}$$

Claim: $\quad c \in \{ x \mid x = \sum \lambda_i a_i \quad \lambda_i \geq 0 \}$



Pf of claim: Suppose not



$d \cdot c > 0$

$d \cdot a_i \leq 0 \quad \forall \; i \in I$

Suppose $d \cdot c > d_0$ & $d \cdot a_i \leq d_0 \; \forall i \in I$
since 0 is in cone $d_0 \geq 0$
if $d_0 > 0$ & $\exists \; d \cdot a_i > 0$ then for λ sufficiently large $d \cdot \lambda a_i > d_0$
$\Rightarrow \forall i \; d \cdot a_i \leq 0$

Then for $\varepsilon > 0$ sufficiently small

$x^* + \varepsilon d$ feasible with $(x^* + \varepsilon d) \cdot c > x^* \cdot c$

$(x^* + \varepsilon d) \cdot a_i < x^* a = b_i \quad$ for $i \in I$

for rest feasibility follows by taking ε small enough

$\rightarrow \leftarrow$

Suppose $\vec{c} = \sum \lambda_i a_i$

Let $y_i = \begin{cases} \lambda_i & i \in I \\ 0 & \text{o.w} \end{cases}$ $\Rightarrow y \geq 0$

$$y \cdot A = c$$

$$y \cdot b = \sum_{i \in I} \lambda_i b_i = \sum_{i \in I} \lambda_i a_i x^* = c \cdot x^*$$

$\boxed{\Sigma}$

Example: max weighted matching

$\max \sum_i \sum_j v_{ij} x_{ij}$ $\min \sum_i u_i + \sum_j p_j$

$\forall i \quad \sum_j x_{ij} \leq 1 \quad u_i$ $u_i + p_j \geq v_{ij} \quad \forall (i,j)$

$\forall j \quad \sum_i x_{ij} \leq 1 \quad p_j$

$x_{ij} \geq 0$ $u_i \geq 0 \quad p_j \geq 0$

$\{x_{ij}\}$ $\{p_j, u_i\}$ feasible $\Rightarrow$ weak duality

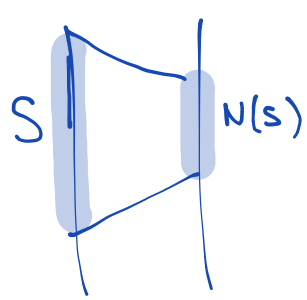
$$\sum_i \sum_j v_{ij} x_{ij} \leq \sum_i \sum_j (u_i + p_j) x_{ij} = \sum_i u_i \sum_j x_{ij} + \sum_j p_j \sum_i x_{ij} \leq \sum_i u_i + \sum_j p_j$$

Duality + Integrality

Let $\{u_i, p_j\}$ be optimal soln to dual. Note: $u_i = \max_j (v_{ij} - p_j)$

Put edge (i,j) if constraint tight $u_i + p_j = v_{ij}$

Claim: $\exists$ perfect matching in graph. If not, by Hall's thm $\exists S$ s.t. $|N(S)| < |S|$

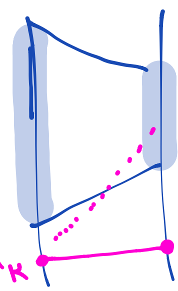


take S maximal such

$\delta = \min_{\substack{i \in S \\ j \notin N(S)}} u_i + p_j - v_{ij}$

$u_i' := u_i - \delta$

$i \in S$



$p_j' = p_j + \delta$

$j \in N(S)$

u_k

$$\sum u_i' + \sum p_j' < \sum u_i + \sum p_j$$

still $u_i := \max_j (v_{ij} - p_j)$

if some u_i' goes -ve, let $\varepsilon = \min p_j$

$$u_i'' := u_i' + \varepsilon \qquad p_j'' := p_j' - \varepsilon$$

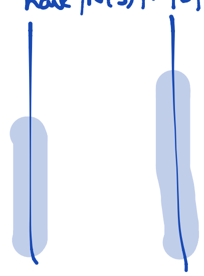
$$\exists\, p_j'' = 0 \wedge u_i + p_j \geq v_{ij} \;\forall (i,j) \Rightarrow u_i \geq 0 \;\forall i$$

$$\Rightarrow \exists \text{ matching s.t. } \sum_{(i,j) \in M} v_{ij} = \sum u_i + \sum p_j$$

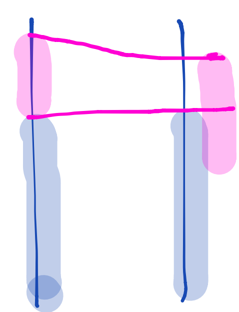
Interpretation: $\exists$ way for each "buyer" to choose one of his favorite items at given prices & for there to be no conflict

Quick proof of Hall's Thm

(1) all subsets have $|N(S)| > |S|$



(2) $\exists$ S s.t. $|N(S)| = |S|$
by induction $\exists$ matching inside



if after applying inductive hypothesis to S, violation $\Rightarrow$ $\exists$ violation to begin with

Interpretation of duality in diet problem

$\min\ c \cdot x$

$Ax \geq b$

$x \geq 0$

c_j : cost/unit of food j

a_{ij} : $\underbrace{\text{amt}}_{\#\text{ units}}$ of vitamin i in food j

b_i : daily unit requirement of vitamin i

x_j : # of units of food j to buy/day

$\max\ b \cdot y$

$y^T A \leq c^T$

$y \geq 0$

y_i : price/unit of vitamin i

$b \cdot y$: revenue of druggist

$\sum_i y_i a_{ij} \leq c_j$ $\equiv$ can't charge more for vitamins than equivalent in food

$y_i \left(\underbrace{\sum_j a_{ij} x_j - b_i}_{>0} \right)$ $\Rightarrow$ vitamin i oversupplied in opt diet

$\underset{CS}{\Rightarrow}$ price for i is 0

$x_j \left(\underbrace{c_j - \sum_i y_i a_{ij}}_{>0} \right)$ $\Rightarrow$ cheaper to buy vitamins than food j

$\underset{CS}{\Rightarrow}$ don't buy food j

Max flow

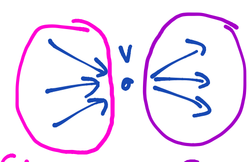
$G=(V,E)$ directed graph w/ capacities $c: E \to \mathbb{R}^+$

two special vertices s, t (assume no edges into s)

Flow $f: E \to \mathbb{R}^+$ satisfies

a) capacity constraints: $\forall e \quad f(e) \le c_e$

b) conservation of flow: $\displaystyle\sum_{e \in \delta^-(v)} f(e) = \sum_{e \in \delta^+(v)} f(e)$



$\delta^+(v)$ edges into v

$\delta^-(v)$ edges out of v

$$\max \sum_{e \in \delta^-(s)} f(e)$$

$$0 \le f(e) \le c_e \qquad e \in E$$

$$\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e) \qquad \forall v \ne s,t$$

max flow LP

An equivalent formulation

$$\max \sum_{p \in \mathcal{P}_{s,t}} f_p$$

$\mathcal{P}_{s,t}$ set of paths from s to t

$$\sum_{p \mid e \in p} f_p \leq c_e \qquad \forall e \qquad y_e$$

$$f_p \geq 0$$

Dual

$$\min \sum c_e y_e$$

$$(*) \qquad \sum_{e \in p} y_e \geq 1 \qquad \forall p \in \mathcal{P}_{s,t}$$

$$y_e \geq 0$$

min fractional cut!

Fix an s-t cut

dual opt $\leq$ min cut

Weak duality $\Rightarrow$ maxflow $\leq$ min cut

Claim that in fact $\exists$ opt integral solution

Thm: Max flow = min cut

Observe: $(*)$ assigns "length" to each edge

length of each s-t path ≥ 1

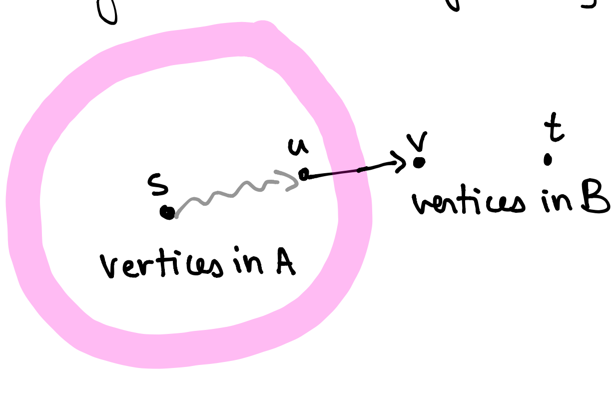
Let $d(v)$ = length of shortest path from $s \rightsquigarrow v$
w.r.t. edge lengths y_e (opt soln to dual)

$d(t) \geq 1$

let $r \sim U(0,1)$

define an s-t cut by letting $A = \{v \mid d(v) \leq r\}$ $\quad s \in A$

$B = \{v \mid d(v) > r\}$ $\quad t \in B$



$$E(\text{capacity of cut}) = \sum_e c_e \underbrace{\Pr(e \text{ cut})}_{= d(v)-d(u) \le y_e}$$

since $d(v) \le d(u) + y_e$

$\Rightarrow \exists$ cut whose capacity $\le$ dual optimal value

Digression on LP algs:

- simplex — not poly time in worst case, but is "smoothed" poly time
- ellipsoid alg
- interior pt methods

} poly time

What does poly time mean? Polynomial in

n # of vars

ϕ : max # bits needed to encode any number

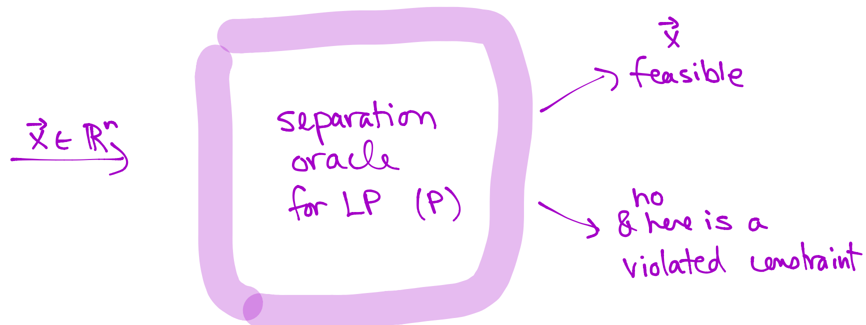
usually m : # of constraints

In fact, sometimes even exp size LPs can be solved in poly time using ellipsoid alg!

Ellipsoid Alg

Alg with remarkable property that can sometimes be used to solve LPs with exponentially many constraints in poly time

if $\exists$ $\boxed{\text{poly time separation oracle}}$



Example: sep oracle for min cut LP we saw : shortest path solver

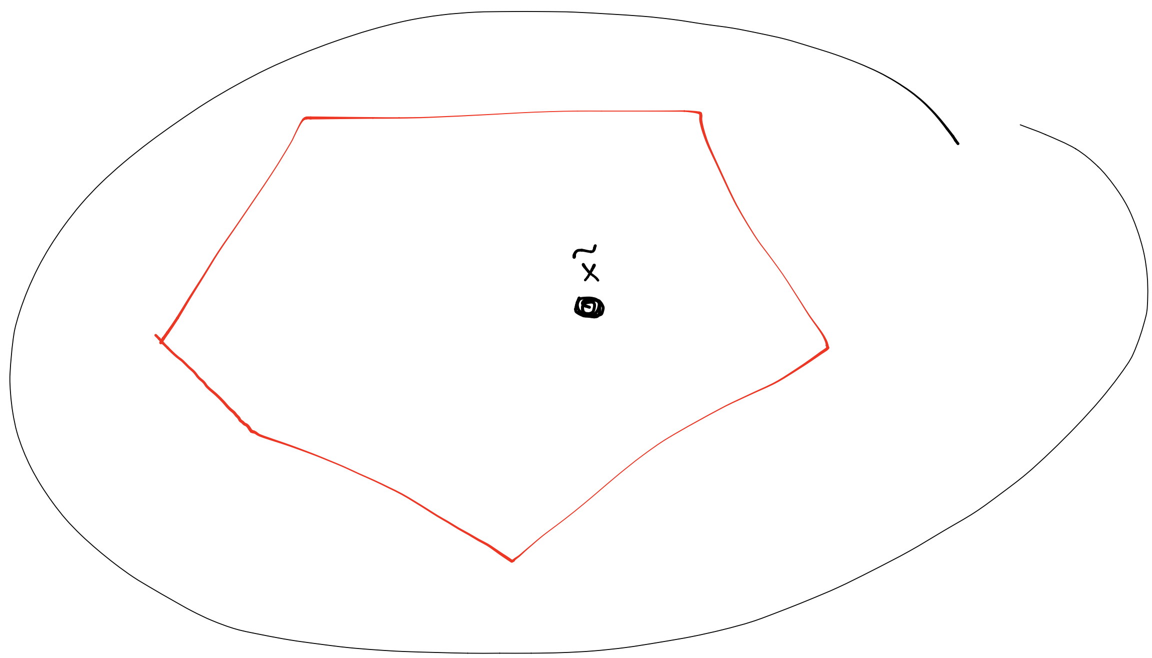
Ellipsoid alg: solves feasibility

$$\max c \cdot x \quad Ax \le b \quad x \ge 0$$

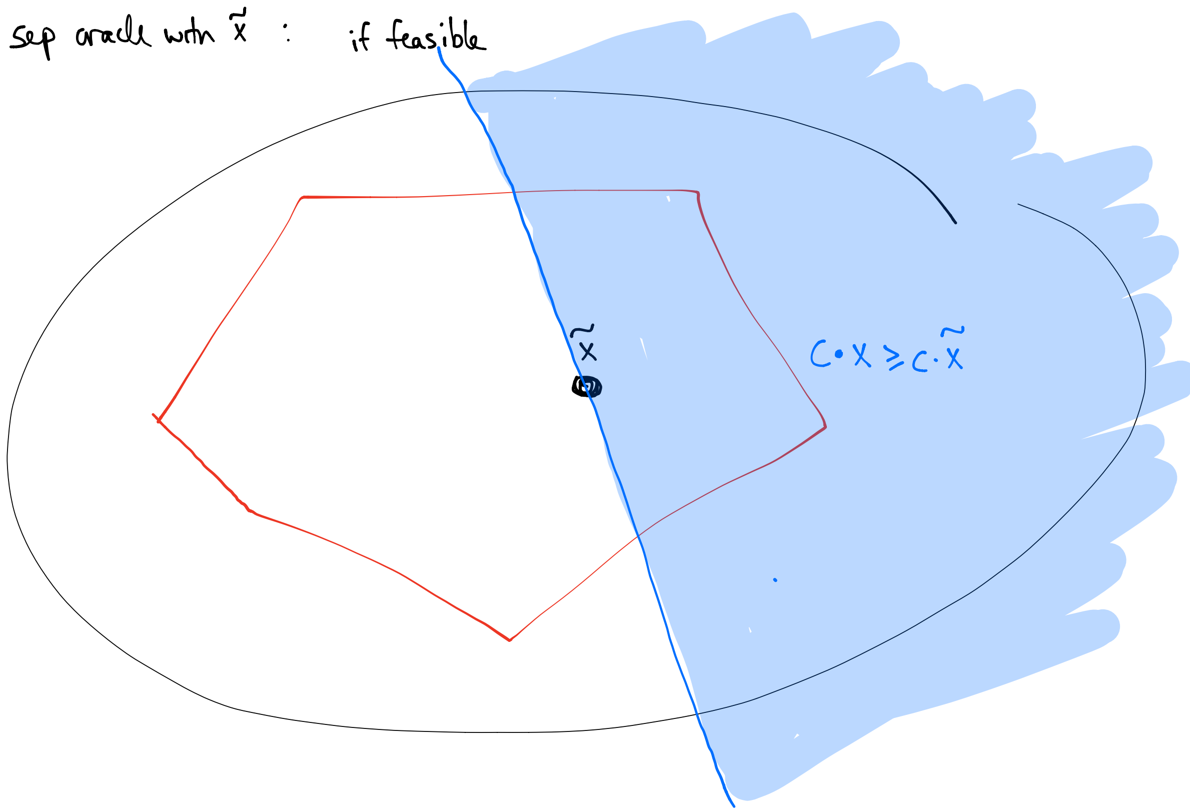
binary search on c_0

$$c \cdot x \ge c_0 \quad A \cdot x \le b \quad x \ge 0$$

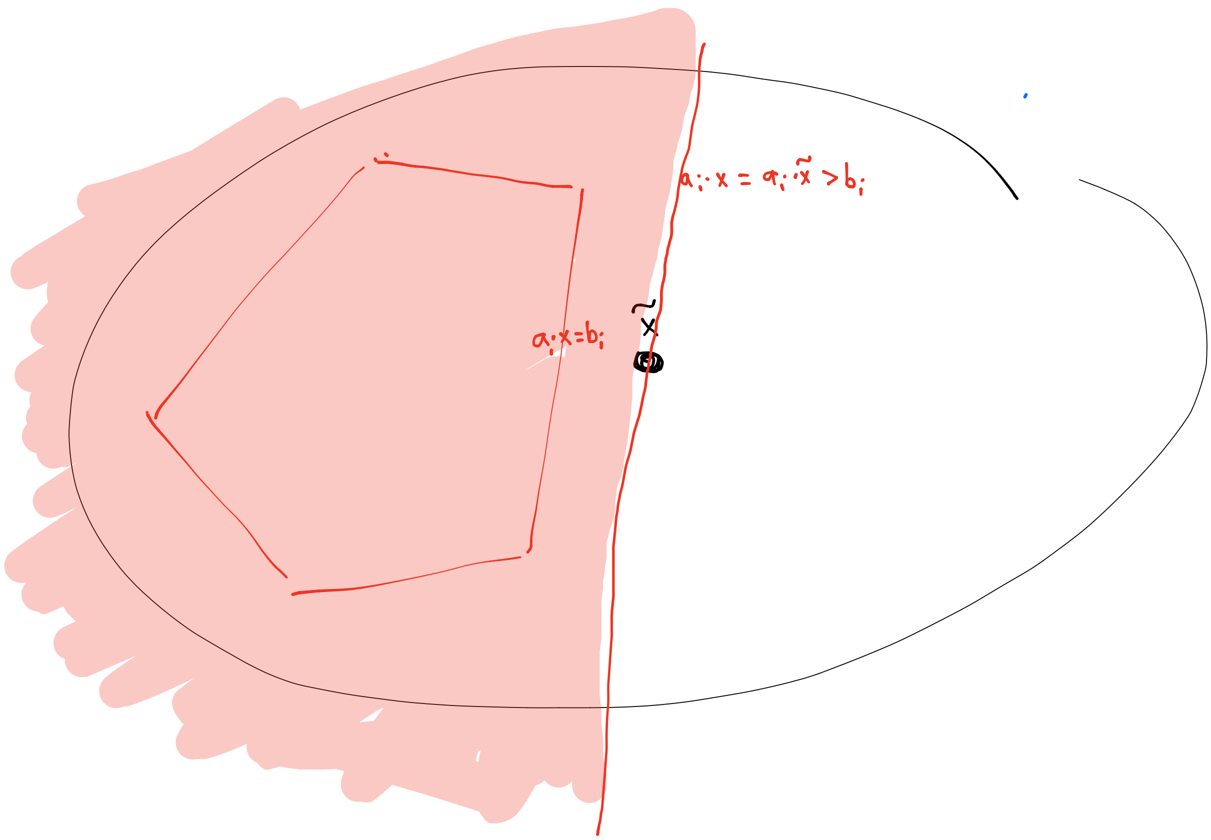
start with ellipsoid containing all vertices of polytope $\tilde{x}$ center of ellipsoid



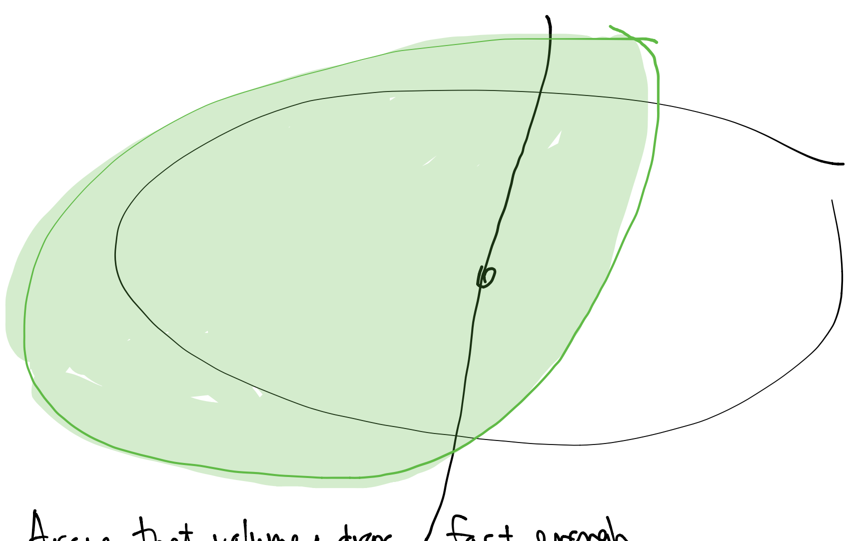
Call sep oracle with $\tilde{x}$: if feasible



if infeasible, with violated constraint $a_i\, x \le b_i$ i.e. $a_i \cdot \tilde{x} > b_i$



in either case $\Rightarrow$ "half" ellipsoid Repeat



Argue that $\underbrace{\text{volume}}_{\text{of ellipsoid}}$ drops fast enough

Other examples

Held-Karp relaxation for TSP (symmetric)

$$\min \sum_e c_e x_e$$

subject to $\displaystyle\sum_{\substack{(u,v) \text{ s.t.} \\ u \in S, v \in \bar{S}}} x_{(u,v)} \geq 2 \quad \forall \emptyset \neq S \neq V$

$$\sum_v x_{(u,v)} = 2 \quad \forall u \in V$$

$$0 \leq x_e \leq 1 \quad \forall e \in E$$

Separation oracle: solve min-cut problem

Survivable network design

design low cost networks that can survive failures

Input: $G=(V,E)$ c_e cost of edge $e \in E$

r_{ij} connectivity requirement for vertices i & j

integer i.e. at least r_{ij} edge disjoint paths from i to j.

integer program

$$\min \sum c_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq \max_{i \in S, j \notin S} r_{ij} \qquad \forall S$$

$$x_e \in \{0,1\}$$

Sep oracle

min i,j cut $\geq r_{ij}$

Solve n^2 max flow problems