CSE 521: Design and Analysis of Algorithms

Assignment #3

due: Tuesday, Oct 25, 10:30AM

Each problem is worth 10 points. KT refers to Algorithm Design, First Edition, by Kleinberg and Tardos. “Give an algorithm” means pseudo-code, a high-level explanation and a proof of correctness. See the website for more grading guidelines.

1. Convergence of the gradient descent algorithm for solving $Ax = b$: Consider the following algorithm:

**Algorithm 1** Gradient descent algorithm for solving linear systems of equations

1. $k \leftarrow 0.$
2. $x_0 \leftarrow 0.$
3. repeat
4. $r_k \leftarrow b - Ax_k$
5. $\alpha_k \leftarrow r_k^T r_k / r_k^T A r_k$.
6. $x_{k+1} \leftarrow x_k + \alpha_k r_k$.
7. $k \leftarrow k + 1.$
8. until $\|\alpha_k\| < 10^{-20}$

(a) Prove that $r_{k+1}^T r_k = 0$ for all $k$.
(b) Define $d_k = A^{-1} b - x_k = A^{-1} r_k$. Define $\delta_k = d_k^T A d_k$ to be a measure of error. Prove that

$$\delta_{k+1} \leq (1 - 1/\kappa) \delta_k.$$

Here $\kappa := \|A\| \cdot \|A^{-1}\|$ is the condition number of $A$, and $\|M\| := \max_{z \neq 0} z^T M z / z^T z$.

2. KT, Chapter 5, Problem 1
3. KT, Chapter 5, Problem 4
4. KT, Chapter 5, Problem 5

5. For two sets $X, Y$ of integers, the Minkowski sum $X + Y$ is the set of all pairwise sums $\{x + y | x \in X, y \in Y\}$. The goal of this problem is to compute $|X + Y|$; that is, the number of elements in $X + Y$. Let $n = |X| = |Y|$ and assume that all elements of $X, Y$ are between 0 and $M$. Further assume that $M$ is small enough so that adds, multiplies, etc of $O(\log M)$-bit numbers takes constant time.

(a) Describe an algorithm to compute $|X + Y|$ in time $O(n^2 \log(n))$.
(b) Describe an algorithm to compute $|X + Y|$ in time $O(M \log(M))$.

(c) For $k$ a positive integer, define $kX = \overbrace{X + X + \cdots + X}^{k \text{ times}}$. Describe an algorithm to compute $|kX|$ in time $O(kM \log(kM))$.

(d) Extra credit: Let $L = |kX|$. Describe a randomized algorithm to compute $|kX|$ with $\geq 2/3$ probability of success in time $O(L^2 \log(L))$. 
