CSE 521: Design and Analysis of Algorithms I

Representative Problems

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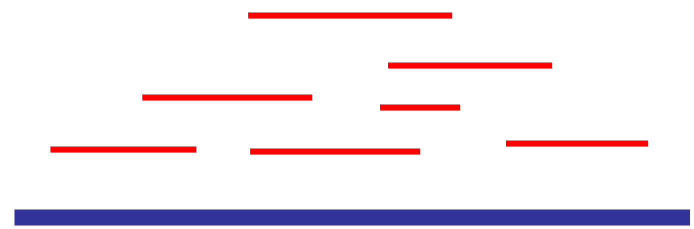
5 Representative Problems

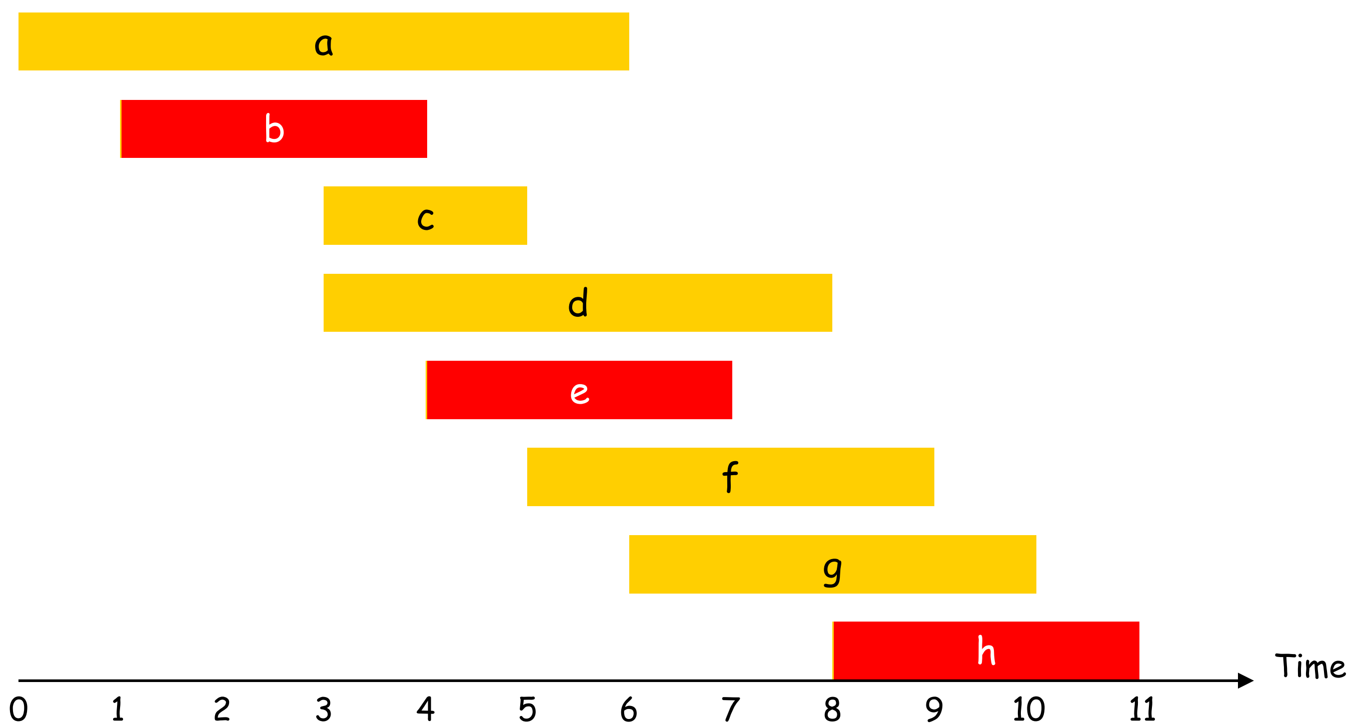
- Interval Scheduling
  - Single resource
  - Reservation requests
    - Of form “Can I reserve it from start time **s** to finish time **f**?”
    - **s** < **f**
  - **Find:** maximum number of requests that can be scheduled so that no two reservations have the resource at the same time



Interval Scheduling

- Input. Set of jobs with start times and finish times.
- Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



Interval scheduling

- Formally
  - Requests **1**,**2**,…,**n**
    - request **i** has start time s_i and finish time $f_i > s_i$
  - Requests **i** and **j** are **compatible** iff either
    - request **i** is for a time entirely before request **j**
      - $f_i \leq s_j$
    - or, request **j** is for a time entirely before request **i**
      - $f_j \leq s_i$
  - Set **A** of requests is **compatible** iff every pair of requests $i,j \in A$, $i \neq j$ is compatible
  - Goal: Find maximum size subset **A** of compatible requests

Interval Scheduling

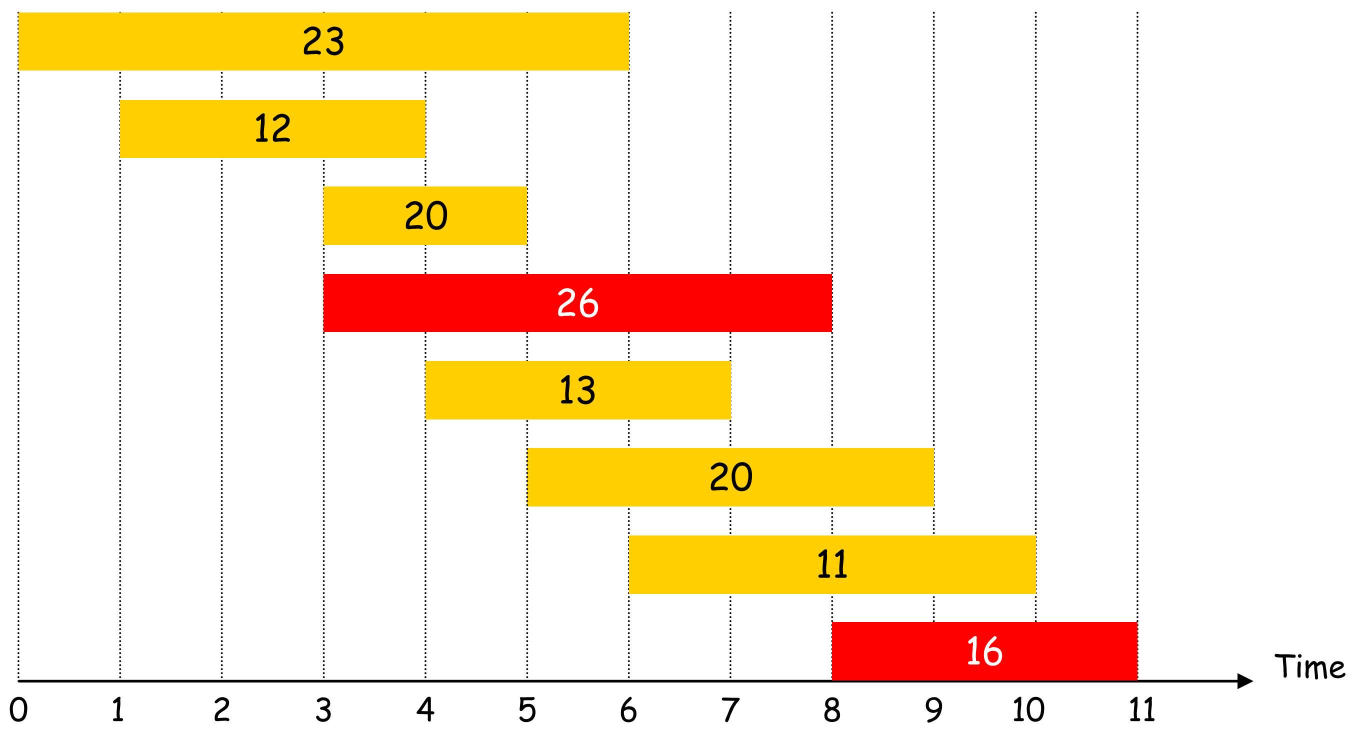
- We shall see that an optimal solution can be found using a “greedy algorithm”
  - Myopic kind of algorithm that seems to have no look-ahead
  - These algorithms only work when the problem has a special kind of structure
  - When they do work they are typically very efficient

Weighted Interval Scheduling

- Same problem as interval scheduling except that each request **i** also has an associated value or weight w_i
  - w_i might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used
- Goal: Find compatible subset **A** of requests with maximum total weight

Weighted Interval Scheduling

- Input. Set of jobs with start times, finish times, and weights.
- Goal. Find maximum weight subset of mutually compatible jobs.



Weighted Interval Scheduling

- Ordinary interval scheduling is a special case of this problem
  - Take all $w_i = 1$
- Problem is quite different though
  - E.g. one weight might dwarf all others
- "Greedy algorithms" don't work

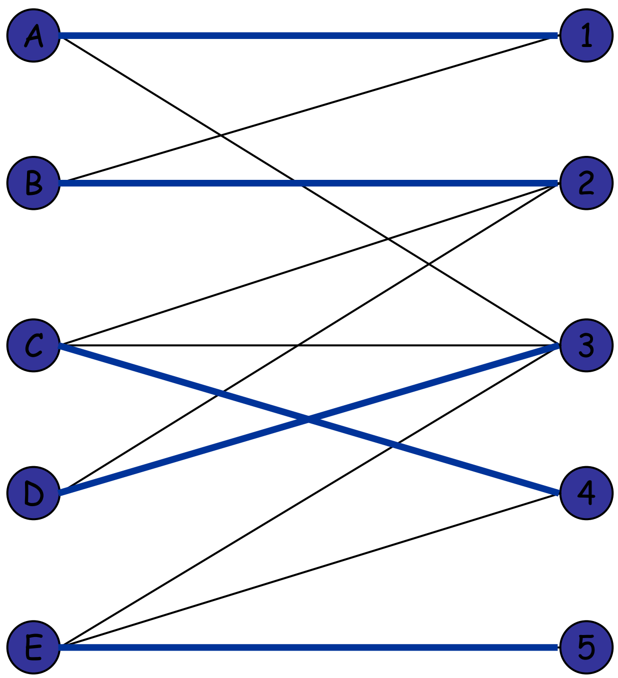
- Solution: "Dynamic Programming"
  - builds up optimal solutions from smaller problems using a compact table to store them

Bipartite Matching

- A graph $G=(V,E)$ is bipartite iff
  - **V** consists of two disjoint pieces **X** and **Y** such that every edge **e** in **E** is of the form **(x,y)** where $x \in X$ and $y \in Y$
  - Similar to stable matching situation but in that case all possible edges were present
- $M \subseteq E$ is a matching in **G** iff no two edges in **M** share a vertex
  - Goal: Find a matching **M** in **G** of maximum possible size

Bipartite Matching

- Input. Bipartite graph.
- Goal. Find maximum cardinality matching.



Bipartite Matching

- Models assignment problems
  - X represents jobs, Y represents machines
  - X represents professors, Y represents courses
- If $|X|=|Y|=n$
  - G has perfect matching iff maximum matching has size n
- **Solution:** polynomial-time algorithm using "augmentation" technique
  - also used for solving more general class of network flow problems

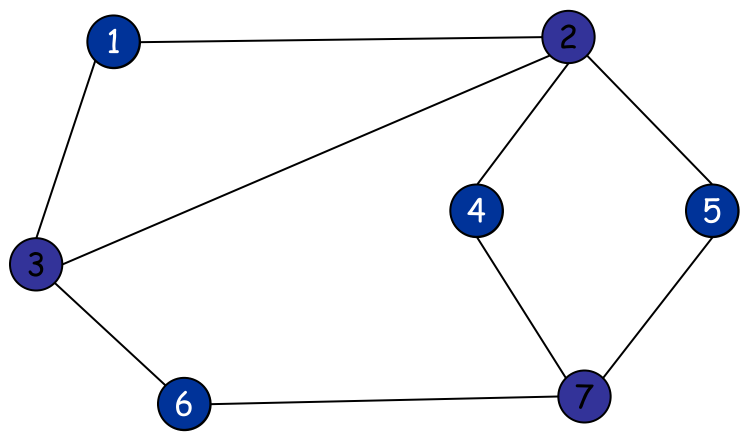
Independent Set

- Given a graph $G=(V,E)$
  - A set $I \subseteq V$ is independent iff no two nodes in I are joined by an edge
- Goal: Find an independent subset I in G of maximum possible size

- Models conflicts and mutual exclusion

Independent Set

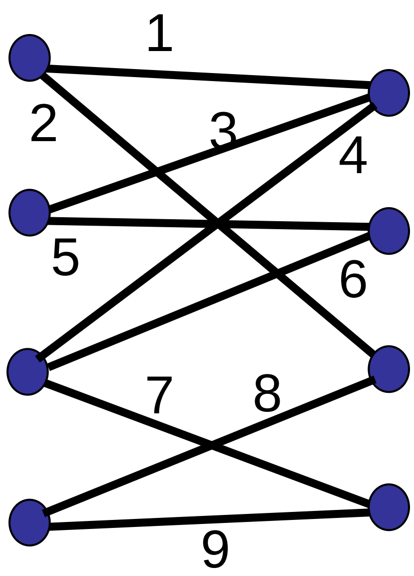
- Input. Graph.
- Goal. Find maximum cardinality independent set.



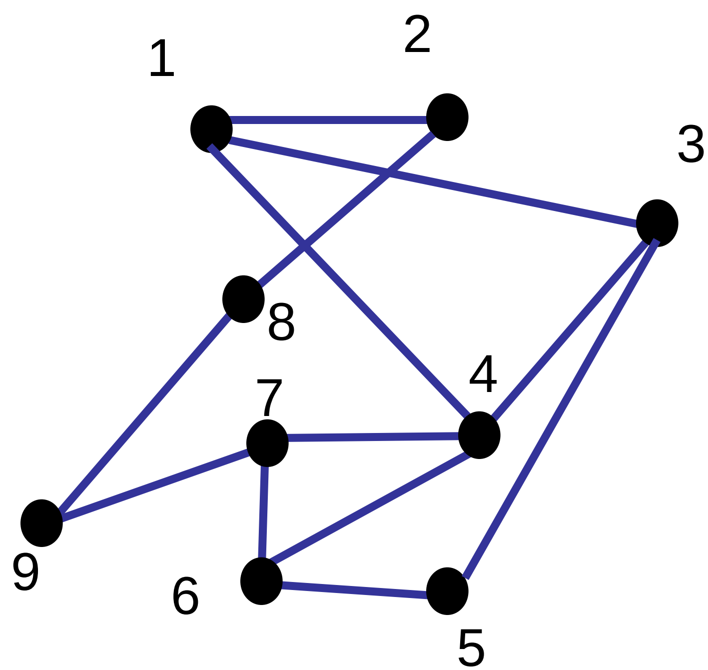
Independent Set

- Generalizes
  - Interval Scheduling
    - Vertices in the graph are the requests
    - Vertices are joined by an edge if they are **not** compatible
  - Bipartite Matching
    - Given bipartite graph **G**=(**V**,**E**) create new graph **G'**=(**V'**,**E'**) where
      - **V'**=**E**
      - Two elements of **V'** (which are edges in **G**) are joined if they share an endpoint in **G**

Bipartite Matching vs Independent Set



G=(U∪V,E)



G’=(V’,E’)

Independent Set

- No polynomial-time algorithm is known
  - But to convince someone that there was a large independent set all you'd need to do is show it to them
    - they can easily convince themselves that the set is large enough and independent
  - Convincing someone that there isn't one seems much harder
- We will show that Independent Set is NP-complete
  - Class of all the hardest problems that have the property above

Competitive Facility Location

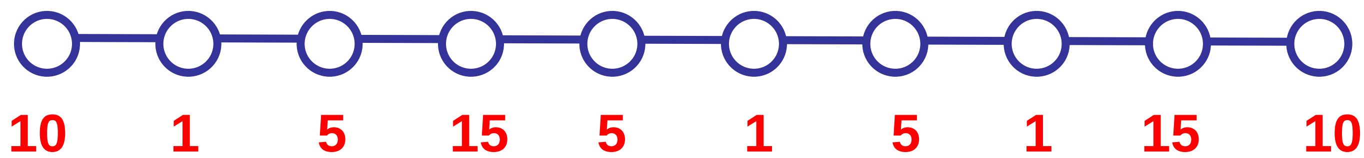
- Two players competing for market share in a geographic area
  - e.g. McDonald's, Burger King
- Rules:
  - Region is divided into **n** zones, **1**,…,**n**
  - Each zone **i** has a value b_i
    - Revenue derived from opening franchise in that zone
  - No adjacent zones may contain a franchise
    - i.e., zoning regulations limit density
  - Players alternate opening franchises
- Find: Given a target total value **B** is there a strategy for the second player that always achieves ≥ **B**?

Competitive Facility Location

- Model geography by
  - A graph $G=(V,E)$ where
    - V is the set $\{1,\dots,n\}$ of zones
    - E is the set of pairs (i,j) such that i and j are adjacent zones

- Observe:
  - The set of zones with franchises will form an independent set in G

Competitive Facility Location



Target **B = 20** achievable ?

What about **B = 25** ?

Competitive Facility Location

- Checking that a strategy is good seems hard
  - You'd have to worry about all possible responses at each round!
    - a giant search tree of possibilities
- Problem is PSPACE-complete
  - Likely strictly harder than NP-complete problems
  - PSPACE-complete problems include
    - Game-playing problems such as **$n \times n$** chess and checkers
    - Logic problems such as whether quantified boolean expressions are always true
    - Verification problems for finite automata

Five Representative Problems

- Variations on a theme: independent set.

- Interval scheduling: n log n greedy algorithm.
- Weighted interval scheduling: n log n dynamic programming algorithm.
- Bipartite matching: n^k max-flow based algorithm.
- Independent set: NP-complete.
- Competitive facility location: PSPACE-complete.