CSE 521: Design & Analysis of Algorithms I

Dealing with NP-completeness

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What to do if the problem you want to solve is NP-hard

- You might have phrased your problem too generally
  - e.g., in practice, the graphs that actually arise are far from arbitrary
    - maybe they have some special characteristic that allows you to solve the problem in your special case
      - for example the Independent-Set problem is easy on “interval graphs”
        - Exactly the case for interval scheduling!
    - search the literature to see if special cases already solved

What to do if the problem you want to solve is NP-hard

- Try to find an approximation algorithm
  - Maybe you can't get the size of the best Vertex Cover but you can find one within a factor of **2** of the best
    - Given graph **G**=(**V**,E), start with an empty cover
    - **While** there are still edges in E left
      - **Choose** an edge **e**={**u**,**v**} in **E** and add both **u** and **v** to the cover
      - Remove all edges from **E** that touch either **u** or **v**.
    - Edges chosen don't share any vertices so optimal cover size must be at least # of edges chosen

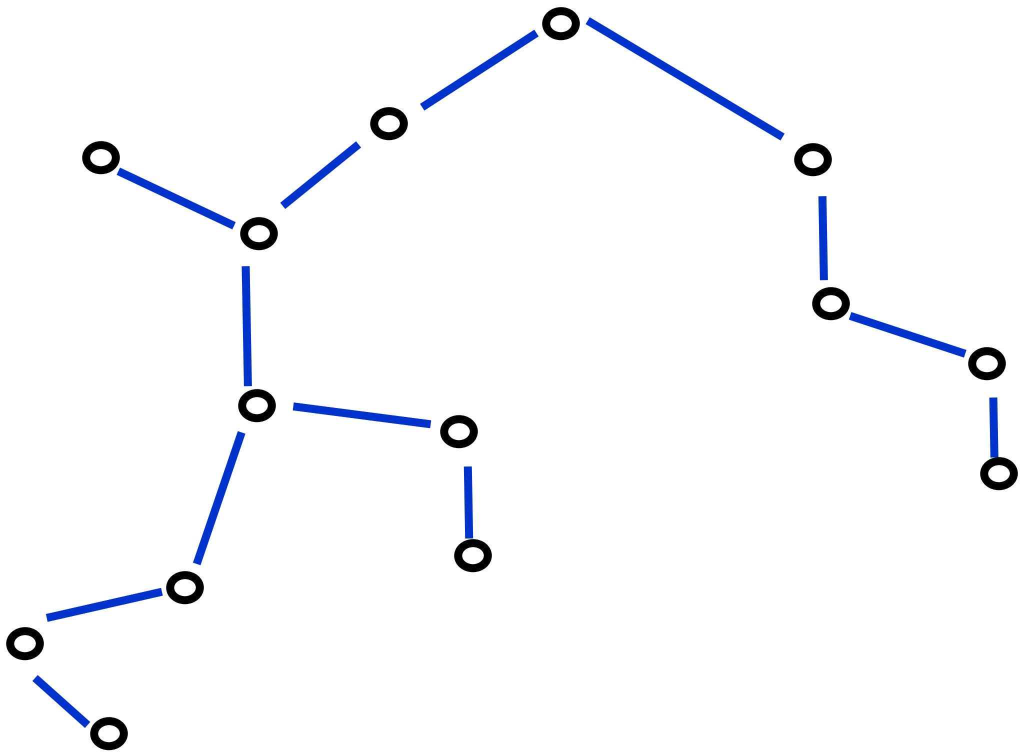
What to do if the problem you want to solve is NP-hard

- Polynomial-time approximation algorithms for **NP**-hard problems can sometimes be ruled out unless **P=NP**
  - E.g. **Coloring Problem:** Given a graph **G=(V,E)** find the smallest **k** such that **G** has a **k**-coloring.
    - No approximation ratio better than **4/3** is possible unless **P=NP**
      - The graph in our **NP**-completeness reduction is always **4**-colorable. This would let us figure out if it is **3**-colorable.

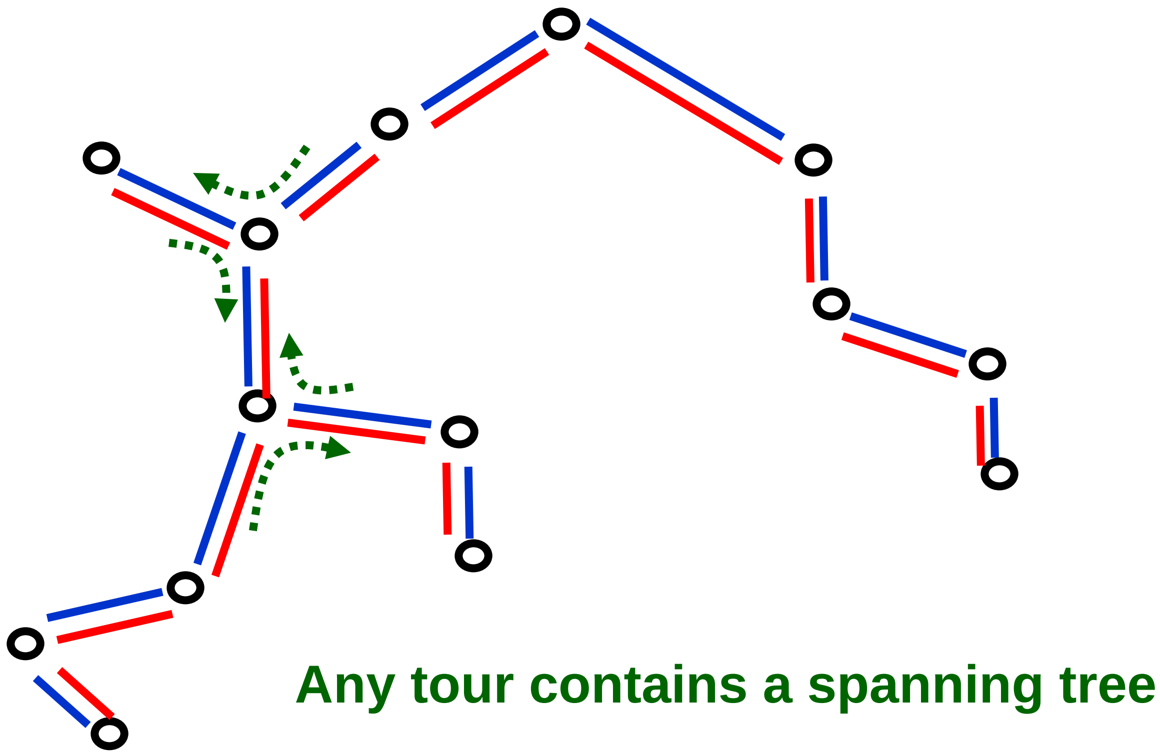
Travelling Salesperson Problem

- TSP
  - Given a weighted graph **G** find of a smallest weight tour that visits all vertices in **G**
- **NP**-hard
- Notoriously easy to obtain close to optimal solutions

Minimum Spanning Tree Approximation



Minimum Spanning Tree Approximation: Factor of 2

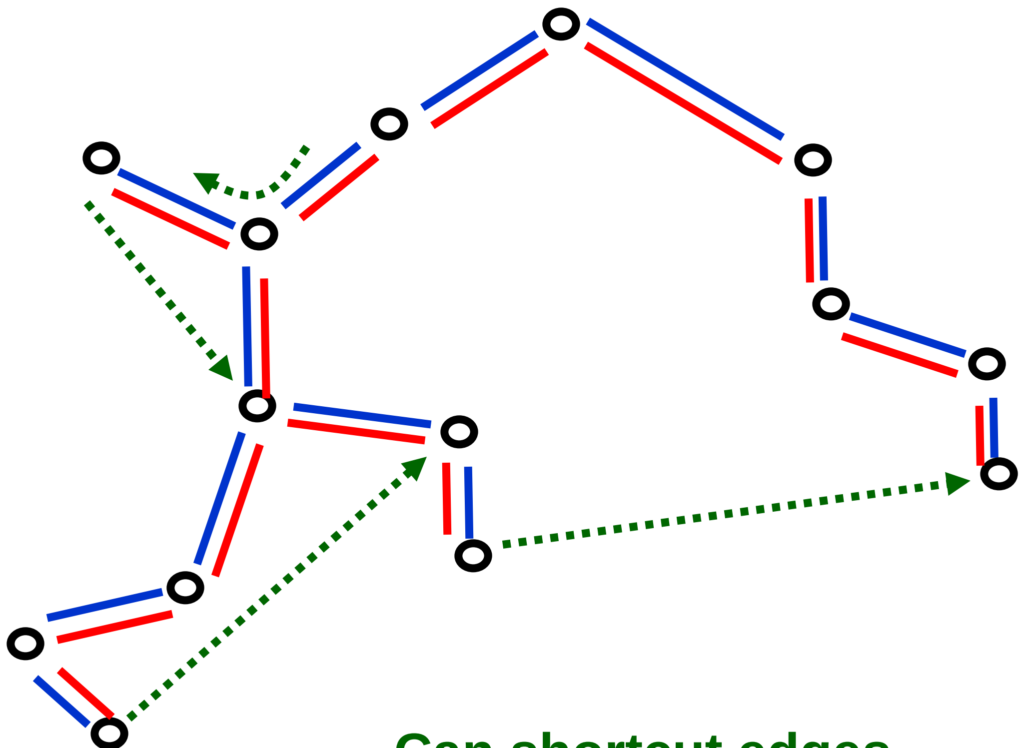


$$\text{MST(G)} \le \text{TOUR}_{\text{OPT}}\text{(G)} \le 2\ \text{MST(G)} \le 2\ \text{TOUR}_{\text{OPT}}\text{(G)}$$

Why did this work?

- We found an Euler tour on a graph that used the edges of the original graph (possibly repeated).
- The weight of the tour was the total weight of the new graph.
- Suppose now
  - All edges possible
  - Weights satisfy triangle inequality
    - **$c(u,w) \leq c(u,v)+c(v,w)$**

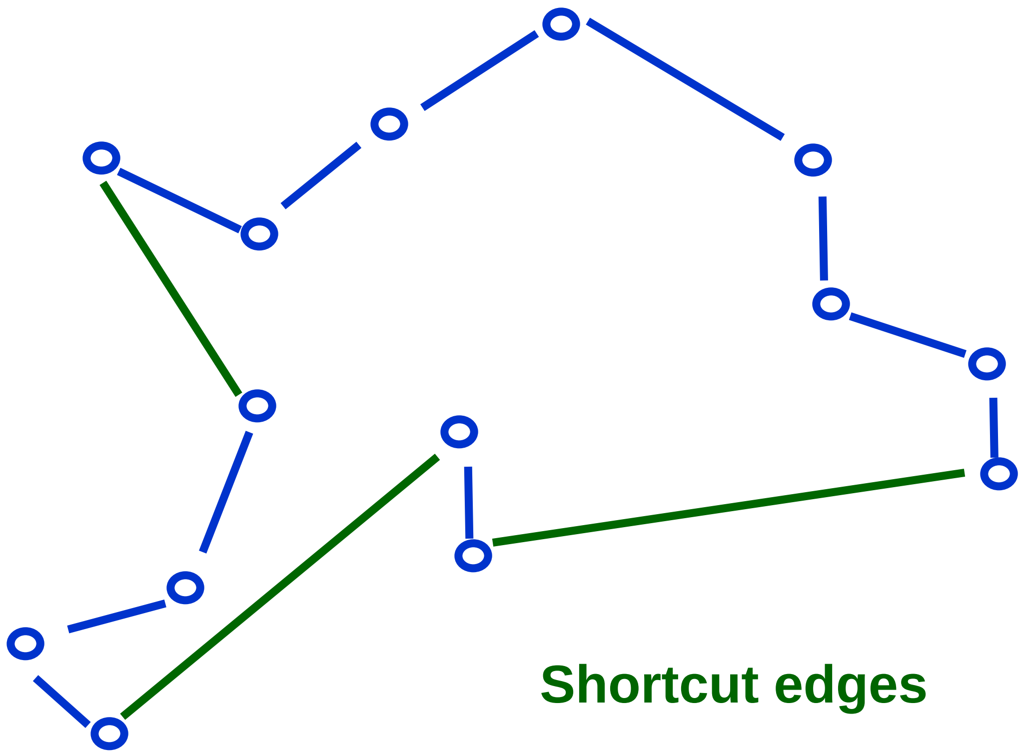
Minimum Spanning Tree Approximation: Triangle Inequality



Can shortcut edges

- Go to next new vertex on the Euler tour

Minimum Spanning Tree Approximation: Factor of 2



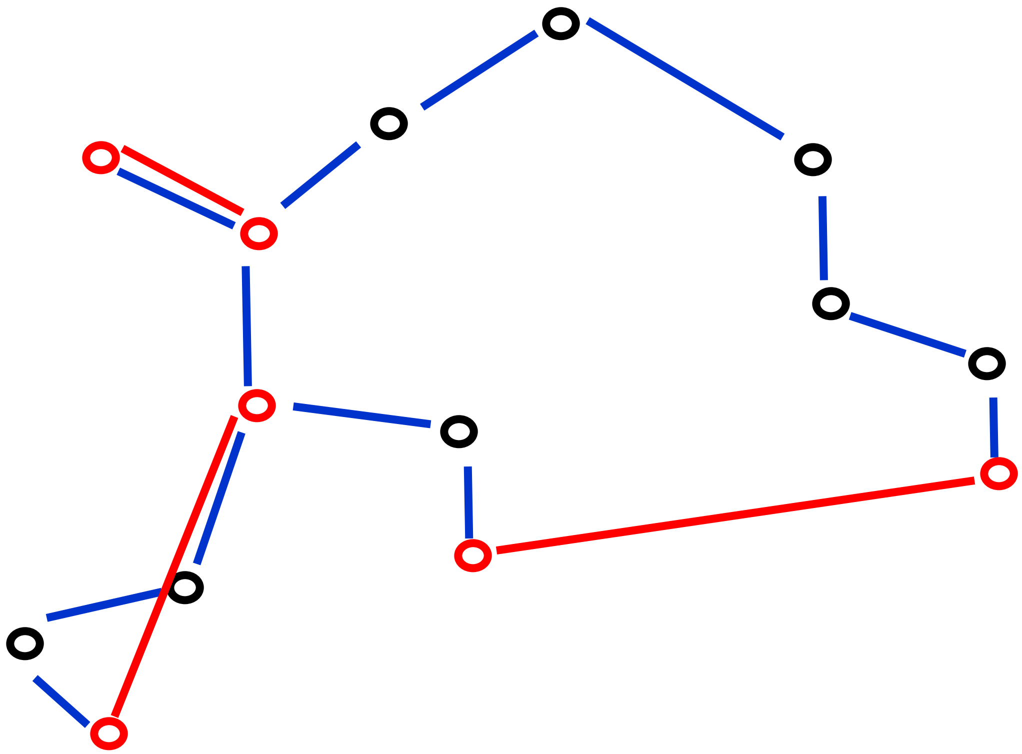
$$\mathbf{TOUR_{OPT}(G) \le 2\ MST(G) \le 2\ TOUR_{OPT}(G)}$$

Christofides Algorithm: A factor 3/2 approximation

- Any Eulerian subgraph of the weighted complete graph will do
  - Eulerian graphs require that all vertices have even degree so

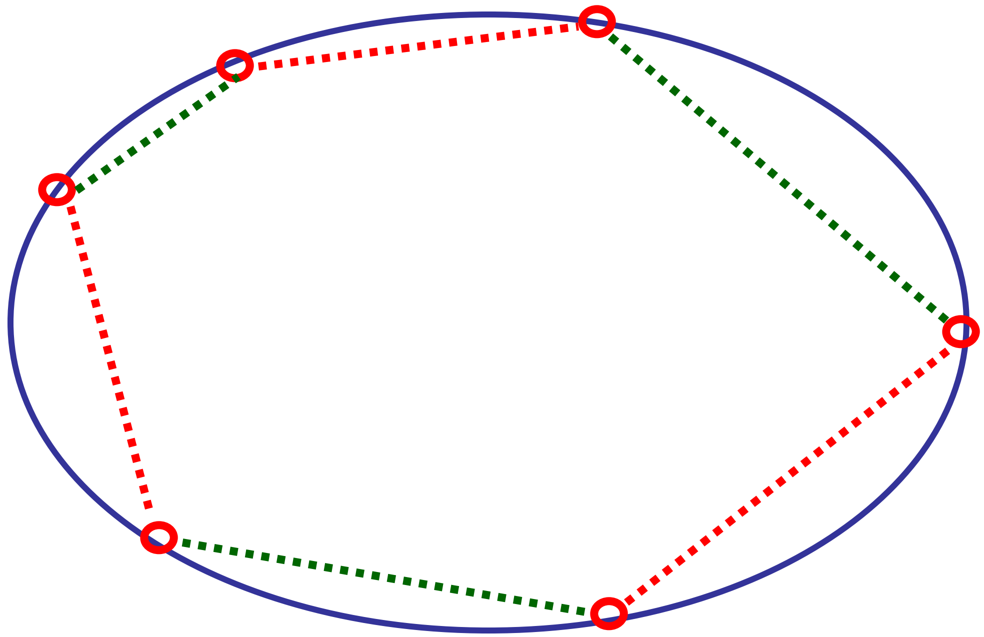
- Christofides Algorithm
  - Compute an MST **T**
  - Find the set **O** of odd-degree vertices in **T**
  - Add a minimum-weight perfect matching **M** on the vertices in **O** to **T** to make every vertex have even degree

Christofides Approximation



Christofides Approximation

Any tour costs at least the cost of two matchings on O



Claim: 2 Cost(M) $\leq$ TOUR$_{OPT}$

Knapsack Problem

- For any $\varepsilon > 0$ can get an algorithm that gets a solution within $(\mathbf{1}+\varepsilon)$ factor of optimal with running time $O(\mathbf{n}^2(1/\varepsilon)^2)$
  - "Polynomial-Time Approximation Scheme" or PTAS
  - Based on maintaining just the high order bits in the dynamic programming solution.

What to do if the problem you want to solve is NP-hard

- More on approximation algorithms
  - Recent research has classified problems based on what kinds of approximations are possible if **P≠NP**
    - Best: **(1+ε)** factor for any **ε>0**.
      - packing and some scheduling problems, TSP in plane
    - Some fixed constant factor > **1**, e.g. **2, 3/2, 100**
      - Vertex Cover, TSP in space, other scheduling problems
    - **Θ(log n)** factor
      - Set Cover, Graph Partitioning problems
    - Worst: **$\Omega(n^{1-\varepsilon})$** factor for any **ε>0**
      - Clique, Independent-Set, Coloring

PCP Theorem and Hardness of Approximation

- PCP (Probabilistically Checkable Proofs) Theorem: Every $A \in NP$ has a polytime verifier V that looks at only 3 random bits of its certificate c such that
  - $x \in A$ $\Rightarrow$ There is a certificate c such that $V(x,c)$ always outputs YES
  - x not $\in A \Rightarrow$ For every certificate c, $V(x,c)$ outputs YES with probability < 0.99999

- Implies that there is a polytime reduction f such that
  - $F \in 3SAT \Rightarrow f(F) \in 3SAT$
  - F not $\in$ #SAT $\Rightarrow$ any truth assignment to $f(F)$ satisfies at most 88% ($< 7/8+\varepsilon$) of clauses of F

What to do if the problem you want to solve is NP-hard

- Try an algorithm that is provably fast "on average".
  - To even try this one needs a model of what a typical instance is.
  - Typically, people consider "random graphs"
    - e.g. all graphs with a given # of edges are equally likely
  - Problems:
    - real data doesn't look like the random graphs
    - distributions of real data aren't analyzable

What to do if the problem you want to solve is NP-hard

- Try to search the space of possible hints/certificates in a more efficient way and hope it is quick enough
  - **Backtracking search**
    - E.g. For SAT there are $\mathbf{2^n}$ possible truth assignments
    - If we set the truth values one-by-one we might be able to figure out whole parts of the space to avoid,
      - e.g. After setting $\mathbf{x_1 \leftarrow 1}$, $\mathbf{x_2 \leftarrow 0}$ we don't even need to set $\mathbf{x_3}$ or $\mathbf{x_4}$ to know that it won't satisfy $(\neg \mathbf{x_1} \vee \mathbf{x}_2) \wedge (\neg \mathbf{x_2} \vee \mathbf{x_3}) \wedge (\mathbf{x_4} \vee \neg \mathbf{x_3}) \wedge (\mathbf{x_1} \vee \neg \mathbf{x_4})$
    - Related technique: **branch-and-bound**
  - Backtracking search can be very effective even with exponential worst-case time
    - For example, the best SAT algorithms used in practice are all variants on backtracking search and can solve surprisingly large problems – more later

What to do if the problem you want to solve is NP-hard

- Use heuristic algorithms and hope they give good answers
  - No guarantees of quality
  - Many different types of heuristic algorithms

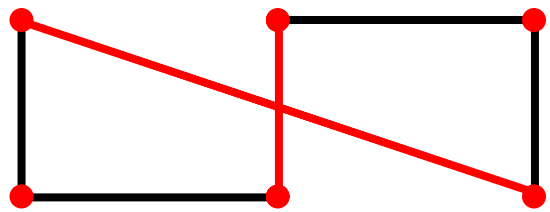
  - Many different options, especially for optimization problems, such as TSP, where we want the best solution.
    - We'll mention several on following slides

Heuristic algorithms for NP-hard problems

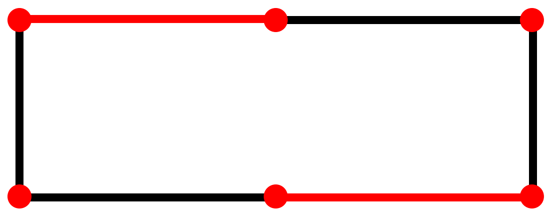
- **local search** for optimization problems
  - need a notion of two solutions being neighbors
  - Start at an arbitrary solution **S**
  - While there is a neighbor **T** of **S** that is better than **S**
    - **S**$\leftarrow$**T**
- Usually fast but often gets stuck in a local optimum and misses the global optimum
  - With some notions of neighbor can take a long time in the worst case

e.g., Neighboring solutions for TSP

Solution S



Solution T



Two solutions are neighbors
iff there is a pair of edges you can
swap to transform one to the other

Heuristic algorithms for NP-hard problems

- **randomized local search**
  - start local search several times from random starting points and take the best answer found from each point
    - **more expensive than plain local search but usually much better answers**
- **simulated annealing**
  - like local search but at each step sometimes move to a worse neighbor with some probability
    - probability of going to a worse neighbor is set to decrease with time as, presumably, solution is closer to optimal
    - helps avoid getting stuck in a local optimum but often **slow to converge** (much more expensive than randomized local search)
    - analogy with slow cooling to get to lowest energy state in a crystal (or in forging a metal)

Heuristic algorithms for NP-hard problems

- **genetic algorithms**
  - view each solution as a **string** (analogy with DNA)
  - maintain a **population of good solutions**
  - allow **random mutations** of single characters of individual solutions
  - **combine two solutions** by taking part of one and part of another (analogy with crossover in sexual reproduction)
  - **go with the winners**: get rid of solutions that have the worst values and make multiple copies of solutions that have the best values (analogy with natural selection -- survival of the fittest).

  - **Not a lot of evidence that they work well**
    - **Often "brittle" when they do – small changes in constraints lead to big changes in solution**
    - **Usually very slow**

  - **However, the "go with the winners" part of the strategy can be combined with local search and works well in that context**

Heuristic algorithms

- **artificial neural networks**
  - based on very elementary model of human neurons
  - **Set up a circuit of artificial neurons**
    - each artificial neuron is an analog circuit gate whose computation depends on a set of **connection strengths**
  - **Train the circuit**
    - Adjust the connection strengths of the neurons by giving many positive & negative training examples and seeing if it behaves correctly
  - **The network is now ready to use**

  - **useful for ill-defined classification problems such as optical character recognition but not typical cut & dried problems**

Other directions

- DNA computing
  - **Each possible hint for an NP problem is represented as a string of DNA**
    - fill a test tube with all possible hints
  - **View verification algorithm as a series of tests**
    - e.g. checking each clause is satisfied in case of Satisfiability
  - **For each test in turn**
    - **use lab operations to filter out all DNA strings that fail the test** (**works in parallel** on all strings; uses PCR)
  - **If any string remains the answer is a YES.**
  - **Relies on fact that Avogadro's # 6×10^{23} is large to get enough strings to fit in a test-tube.**
  - **Error-prone & problem sizes typically very small!**

Other directions

- Quantum computing
  - **Use physical processes at the quantum level to implement "weird" kinds of circuit gates**
    - unitary transformations
  - **Quantum objects can be in a superposition of many pure states at once**
    - can have **n** objects together in a superposition of 2^n states
  - **Each quantum circuit gate operates on the whole superposition of states at once**
    - inherent **parallelism** but classical randomized algorithms have a similar parallelism: **not enough on its own**
    - **Advantage over classical: parallel copies interfere with each other**
    - **Can reduce brute force search from 2^n to $2^{n/2}$ time**

  - **Strong evidence that they won't solve NP-complete problems efficiently**

  - **Theoretically able to factor efficiently.**

  - **Large practical problems: errors, decoherence that need to be overcome.**