Natural Language Processing (CSE 517): Language Models

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Very Quick Review of Probability

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  $p(X = x, Y = y) = p(X = x \mid Y = y) \cdot p(Y = y) = p(Y = y \mid X = x) \cdot p(X = x)$

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- Sometimes true: $p(X = x, Y = y) = p(X = x) \cdot p(Y = y)$
- The difference between *true* and *estimated* probability distributions

Language Models: Definitions

- $\mathcal{V}$ is a finite set of (discrete) symbols (☺ "words" or possibly characters); $V = |\mathcal{V}|$
- $\mathcal{V}^\dagger$ is the (infinite) set of sequences of symbols from $\mathcal{V}$ whose final symbol is ⯃
- $p : \mathcal{V}^\dagger \rightarrow \mathbb{R}$, such that:
  - For any $\boldsymbol{x} \in \mathcal{V}^\dagger$, $p(\boldsymbol{x}) \geq 0$
  - $\displaystyle\sum_{\boldsymbol{x} \in \mathcal{V}^\dagger} p(\boldsymbol{X} = \boldsymbol{x}) = 1$

  (I.e., p is a proper probability distribution.)

Language modeling: estimate p from examples, $\boldsymbol{x}_{1:n} = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$.

Immediate Objections

1. Why would we want to do this?
2. Are the nonnegativity and sum-to-one constraints really necessary?
3. Is "finite $\mathcal{V}$" realistic?

Motivation: Noisy Channel Models

A pattern for modeling a pair of random variables, $\boldsymbol{D}$ and $\boldsymbol{O}$:

$$\boxed{\text{source}} \longrightarrow \boldsymbol{D} \longrightarrow \boxed{\text{channel}} \longrightarrow \boldsymbol{O}$$

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- $\boldsymbol{D}$ is the plaintext, the true message, the missing information, the output
- $\boldsymbol{O}$ is the ciphertext, the garbled message, the observable evidence, the input
- Decoding: select $\boldsymbol{d}$ given $\boldsymbol{O} = \boldsymbol{o}$.

$$\begin{aligned}
\boldsymbol{d}^* &= \operatorname*{argmax}_{\boldsymbol{d}} p(\boldsymbol{d} \mid \boldsymbol{o}) \\
&= \operatorname*{argmax}_{\boldsymbol{d}} \frac{p(\boldsymbol{o} \mid \boldsymbol{d}) \cdot p(\boldsymbol{d})}{p(\boldsymbol{o})} \\
&= \operatorname*{argmax}_{\boldsymbol{d}} \underbrace{p(\boldsymbol{o} \mid \boldsymbol{d})}_{\text{channel model}} \cdot \underbrace{p(\boldsymbol{d})}_{\text{source model}}
\end{aligned}$$

Noisy Channel Example: Speech Recognition

$$\boxed{\text{source}} \longrightarrow \text{sequence in } \mathcal{V}^\dagger \longrightarrow \boxed{\text{channel}} \longrightarrow \text{acoustics}$$

- Acoustic model defines $p(\text{sounds} \mid \boldsymbol{d})$ (channel)
- Language model defines $p(\boldsymbol{d})$ (source)

Noisy Channel Example: Speech Recognition

Credit: Luke Zettlemoyer

word sequence	$\log p(\text{acoustics} \mid \text{word sequence})$
the station signs are in deep in english	-14732
the stations signs are in deep in english	-14735
the station signs are in deep into english	-14739
the station 's signs are in deep in english	-14740
the station signs are in deep in the english	-14741
the station signs are indeed in english	-14757
the station 's signs are indeed in english	-14760
the station signs are indians in english	-14790
the station signs are indian in english	-14799
the stations signs are indians in english	-14807
the stations signs are indians and english	-14815

Noisy Channel Example: Machine Translation

Also knowing nothing official about, but having guessed and inferred considerable about, the powerful new mechanized methods in cryptography—methods which I believe succeed even when one does not know what language has been coded—one naturally wonders if the problem of translation could conceivably be treated as a problem in cryptography. When I look at an article in Russian, I say: "This is really written in English, but it has been coded in some strange symbols. I will now proceed to decode."

Warren Weaver, 1955

Noisy Channel Examples

- Speech recognition
- Machine translation
- Optical character recognition
- Spelling and grammar correction

“Conditional” Language Models

Instead of $p(\boldsymbol{X})$, model $p(\boldsymbol{X} \mid Context)$.

- $Context$ could be an input (acoustics, source-language sentence, image of text)
  . . . or it could be something else (visual input, stock prices, . . .)
- Made possible by advances in machine learning!

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Evaluation: Perplexity

Intuitively, language models should assign high probability to real language they have not seen before.

For out-of-sample ("held-out" or "test") data $\bar{\boldsymbol{x}}_{1:m}$:

- Probability of $\bar{\boldsymbol{x}}_{1:m}$ is $\prod_{i=1}^{m} p(\bar{\boldsymbol{x}}_i)$

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- Average log-probability per word of $\bar{\boldsymbol{x}}_{1:m}$ is

$$l = \frac{1}{M} \sum_{i=1}^{m} \log_2 p(\bar{\boldsymbol{x}}_i)$$

if $M = \sum_{i=1}^{m} |\bar{\boldsymbol{x}}_i|$ (total number of words in the corpus)

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- Perplexity (relative to $\bar{\boldsymbol{x}}_{1:m}$) is 2^{-l}

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Lower is better.

Understanding Perplexity

$$2^{-\frac{1}{M}\sum_{i=1}^{m}\log_2 p(\bar{\boldsymbol{x}}_i)}$$

It's a branching factor!

- Assign probability of 1 to the test data $\Rightarrow$ perplexity $= 1$
- Assign probability of $\frac{1}{|\mathcal{V}|}$ to every word $\Rightarrow$ perplexity $= |\mathcal{V}|$
- Assign probability of 0 to *anything* $\Rightarrow$ perplexity $= \infty$
  - This motivates a stricter constraint than we had before:
    - For any $\boldsymbol{x} \in \mathcal{V}^\dagger$, $p(\boldsymbol{x}) > 0$

Perplexity

- Perplexity on conventionally accepted test sets is often reported in papers.
- Generally, I won't discuss perplexity numbers much, because:
  - Perplexity is only an intermediate measure of performance.
  - Understanding the models is more important than remembering how well they perform on particular train/test sets.
- If you're curious, look up numbers in the literature; always take them with a grain of salt!

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Is “finite $\mathcal{V}$” realistic?

No

Is “finite $\mathcal{V}$” realistic?

No

no

n0

-no

notta

Nº

/no

//no

(no

|no

The Language Modeling Problem

Input: $\boldsymbol{x}_{1:n}$ ("training data")
Output: $p : \mathcal{V}^{\dagger} \rightarrow \mathbb{R}^{+}$
☺ p should be a "useful" measure of plausibility (not grammaticality).

A Trivial Language Model

$$p(\boldsymbol{x}) = \frac{|\{i \mid \boldsymbol{x}_i = \boldsymbol{x}\}|}{n} \qquad = \frac{c_{\boldsymbol{x}_{1:n}}(\boldsymbol{x})}{n}$$

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What if $\boldsymbol{x}$ is not in the training data?

Using the Chain Rule

$$
p(\boldsymbol{X} = \boldsymbol{x}) = \left(\begin{array}{l}
p(X_1 = x_1 \mid X_0 = x_0) \\
\cdot\ p(X_2 = x_2 \mid X_{0:1} = x_{0:1}) \\
\cdot\ p(X_3 = x_3 \mid X_{0:2} = x_{0:2}) \\
\vdots \\
\cdot\ p(X_\ell = {\color{red}\bigcirc} \mid X_{0:\ell-1} = x_{0:\ell-1})
\end{array} \right)
$$

$$
= \prod_{j=1}^{\ell} p(X_j = x_j \mid X_{0:j-1} = x_{0:j-1})
$$

Unigram Model

$$p(\boldsymbol{X} = \boldsymbol{x}) = \prod_{j=1}^{\ell} p(X_j = x_j \mid X_{0:j-1} = x_{0:j-1})$$

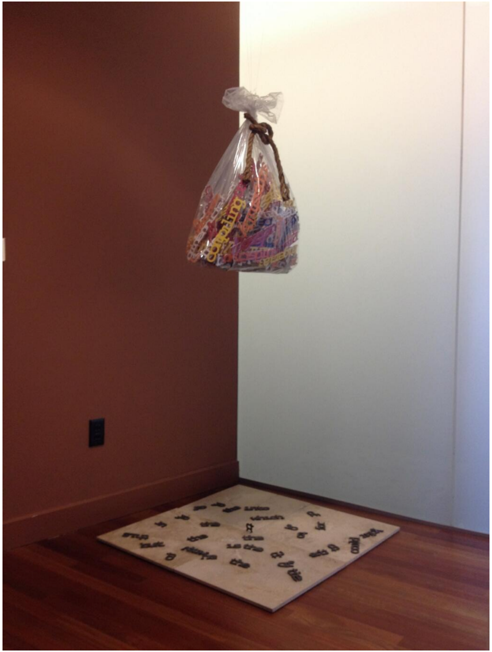
$$\overset{\text{assumption}}{=} \prod_{j=1}^{\ell} p_{\boldsymbol{\theta}}(X_j = x_j) = \prod_{j=1}^{\ell} \theta_{x_j} \approx \prod_{j=1}^{\ell} \hat{\theta}_{x_j}$$

Maximum likelihood estimate:

$$\forall v \in \mathcal{V}, \hat{\theta}_v = \frac{|\{i, j \mid [\boldsymbol{x}_i]_j = v\}|}{N} = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

where $N = \sum_{i=1}^{n} |\boldsymbol{x}_i|$.

Also known as "relative frequency estimation."



Unigram Model

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Maximum likelihood estimate:

$$\forall v \in \mathcal{V}, \hat{\theta}_v = \frac{|\{i, j \mid [\boldsymbol{x}_i]_j = v\}|}{N}$$

$$= \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

where $N = \sum_{i=1}^{n} |\boldsymbol{x}_i|$.
Also known as "relative frequency estimation."

Relative Frequency Estimation is the MLE

(Unigram Model)

The maximum likelihood estimation problem:

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n})$$

Relative Frequency Estimation is the MLE

(Unigram Model)

Logarithm is a monotonic function.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n}) = \exp \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n})$$

Relative Frequency Estimation is the MLE
(Unigram Model)

Each sequence is an independent sample from the model.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n}) = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

Relative Frequency Estimation is the MLE
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Plug in the form of the unigram model.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\boldsymbol{x}_i) = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} \theta_{[\boldsymbol{x}_i]_j}$$

Relative Frequency Estimation is the MLE
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Log of product equals sum of logs.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} \theta_{[\boldsymbol{x}_i]_j} = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \log \theta_{[\boldsymbol{x}_i]_j}$$

Relative Frequency Estimation is the MLE

(Unigram Model)

Convert from tokens to types.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \log \theta_{[\boldsymbol{x}_i]_j} = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v$$

Relative Frequency Estimation is the MLE
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Convert to a minimization problem (for consistency with textbooks).

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v = \min_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} -\sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v$$

Relative Frequency Estimation is the MLE

(Unigram Model)

Lagrange multiplier to convert to a less constrained problem.

$$\min_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v$$

$$= \max_{\mu \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

$$= \min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

Intuitively, if $\sum_{v \in \mathcal{V}} \theta_v$ gets too big, μ will push toward $+\infty$.

For more about Lagrange multipliers, see Dan Klein's tutorial (reference at the end of these slides).

Relative Frequency Estimation is the MLE

(Unigram Model)

Use first-order conditions to solve for $\boldsymbol{\theta}$ in terms of μ.

$$\min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

$$\text{fixing } \mu \text{, for all } v \text{, set: } 0 = \frac{\partial}{\partial \theta_v}$$

$$= \frac{-c_{\boldsymbol{x}_{1:n}}(v)}{\theta_v} + \mu$$

$$\theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}$$

Relative Frequency Estimation is the MLE

(Unigram Model)

Plug in for each θ_v.

$$\min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

$$= \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} - \mu \left(1 - \sum_{v \in \mathcal{V}} \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} \right)$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

Relative Frequency Estimation is the MLE

(Unigram Model)

Rearrange terms ($a \log \frac{a}{b} = a \log a - a \log b$ and $N = \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v)$).

$$\max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} - \mu \left(1 - \sum_{v \in \mathcal{V}} \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} \right)$$

$$= \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

Relative Frequency Estimation is the MLE

(Unigram Model)

Use first-order conditions to solve for μ.

$$\max_{\mu \geq 0} -\sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

$$\begin{aligned} \text{set: } 0 &= \frac{\partial}{\partial \mu} \\ &= \frac{N}{\mu} - 1 \\ \mu &= N \end{aligned}$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

Relative Frequency Estimation is the MLE

(Unigram Model)

Plug in for μ.

$$\max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

$$= - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log N$$

$$\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}} = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

… and that's the relative frequency estimate!

Unigram Models: Assessment

Pros:

- Easy to understand
- Cheap
- Good enough for information retrieval (maybe)

Cons:

- "Bag of words" assumption is linguistically inaccurate
  - $p(\text{the the the the}) \gg p(\text{I want ice cream})$
- Data sparseness; high variance in the estimator
- "Out of vocabulary" problem

Markov Models $\equiv$ n-gram Models

$$p(\boldsymbol{X} = \boldsymbol{x}) = \prod_{j=1}^{\ell} p(X_j = x_j \mid X_{0:j-1} = x_{0:j-1})$$

$$\stackrel{\text{assumption}}{=} \prod_{j=1}^{\ell} p_{\boldsymbol{\theta}}(X_j = x_j \mid X_{j-\mathsf{n}+1:j-1} = x_{j-\mathsf{n}+1:j-1})$$

$(\mathsf{n}-1)$th-order Markov assumption $\equiv$ n-gram model

- Unigram model is the $\mathsf{n} = 1$ case
- For a long time, trigram models ($\mathsf{n} = 3$) were widely used
- 5-gram models ($\mathsf{n} = 5$) were common in MT for a time

Estimating n-Gram Models

	unigram	bigram	trigram
$p_{\boldsymbol{\theta}}(\boldsymbol{x}) =$	$\prod_{j=1}^{\ell} \theta_{x_j}$	$\prod_{j=1}^{\ell} \theta_{x_j \mid x_{j-1}}$	$\prod_{j=1}^{\ell} \theta_{x_j \mid x_{j-2} x_{j-1}}$
Parameters:	θ_v $\forall v \in \mathcal{V}$	$\theta_{v \mid v'}$ $\forall v \in \mathcal{V}, v' \in \mathcal{V} \cup \{\bigcirc\}$	$\theta_{v \mid v'' v'}$ $\forall v \in \mathcal{V}, v', v'' \in \mathcal{V} \cup \{\bigcirc\}$
MLE:	$\frac{c(v)}{N}$	$\frac{c(v'v)}{\sum_{u \in \mathcal{V}} c(v'u)}$	$\frac{c(v''v'v)}{\sum_{u \in \mathcal{V}} c(v''v'u)}$

General case:

$$\prod_{j=1}^{\ell} \theta_{x_j \mid \boldsymbol{x}_{j-\mathsf{n}+1:j-1}} \qquad \theta_{v \mid \boldsymbol{h}}, \ \forall v \in \mathcal{V}, \boldsymbol{h} \in (\mathcal{V} \cup \{\bigcirc\})^{\mathsf{n}-1} \qquad \frac{c(\boldsymbol{h}v)}{\sum_{u \in \mathcal{V}} c(\boldsymbol{h}u)}$$

The Problem with MLE

- The curse of dimensionality: the number of parameters grows exponentially in n
- Data sparseness: most n-grams will never be observed, even if they are linguistically plausible
- No one actually uses the MLE!

Smoothing

A few years ago, I'd have spent a whole lecture on this! ☹

- Simple method: add $\lambda > 0$ to every count (including zero-counts) before normalizing
- What makes it hard: ensuring that the probabilities over all sequences sum to one
  - Otherwise, perplexity calculations break
- Longstanding champion: modified Kneser-Ney smoothing (Chen and Goodman, 1998)
- Stupid backoff: reasonable, easy solution when you don't care about perplexity (Brants et al., 2007)

References I

Thorsten Brants, Ashok C. Popat, Peng Xu, Franz J. Och, and Jeffrey Dean. Large language models in machine translation. In *Proc. of EMNLP-CoNLL*, 2007.

Stanley F. Chen and Joshua Goodman. An empirical study of smoothing techniques for language modeling. Technical Report TR-10-98, Center for Research in Computing Technology, Harvard University, 1998.

Dan Klein. Lagrange multipliers without permanent scarring, Undated. URL `https://www.cs.berkeley.edu/~klein/papers/lagrange-multipliers.pdf`.