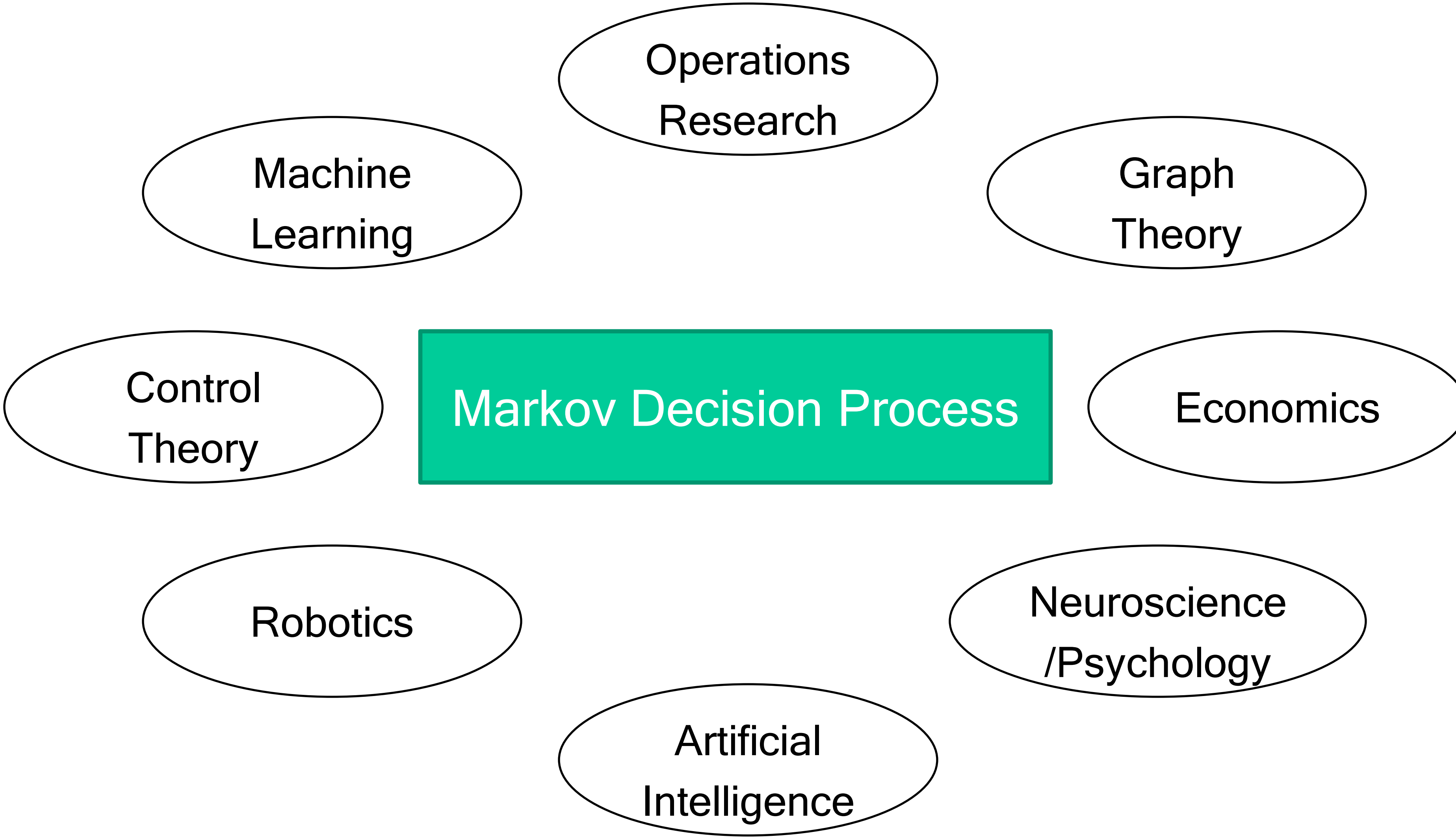
Markov Decision Processes

Mausam

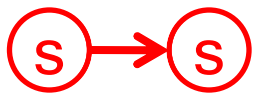
CSE 515



model the sequential decision making of a rational agent.

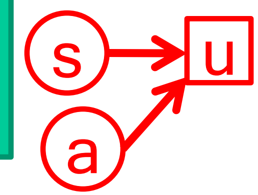
A Statistician's view to MDPs

Markov Chain



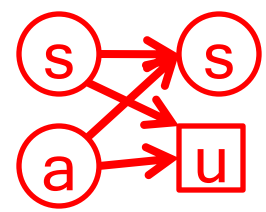
- sequential process
- models state transitions
- autonomous process

One-step Decision Theory



- one-step process
- models choice
- maximizes utility

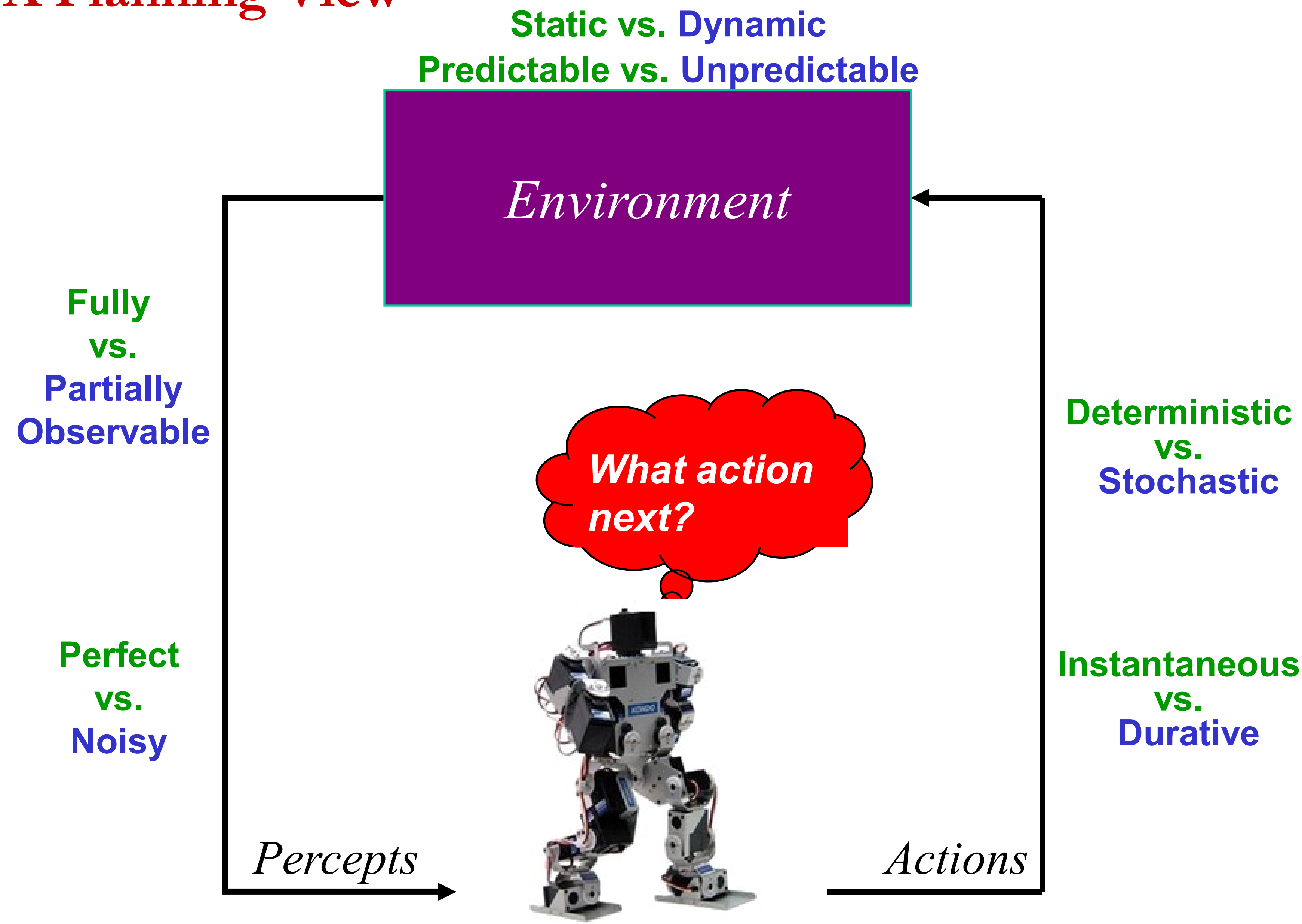
Markov Decision Process



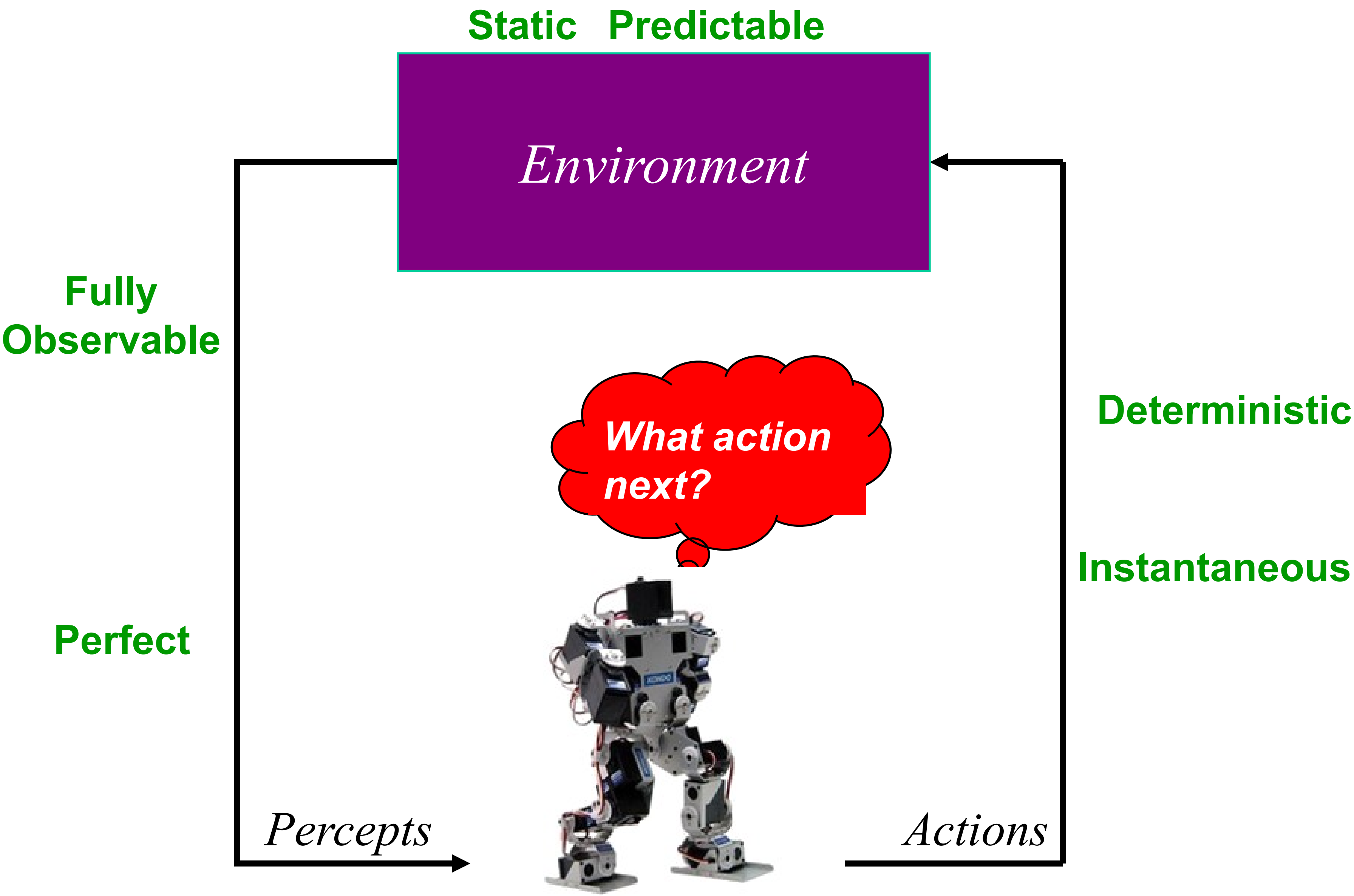
- Markov chain + choice
- Decision theory + sequentiality

- sequential process
- models state transitions
- models choice
- maximizes utility

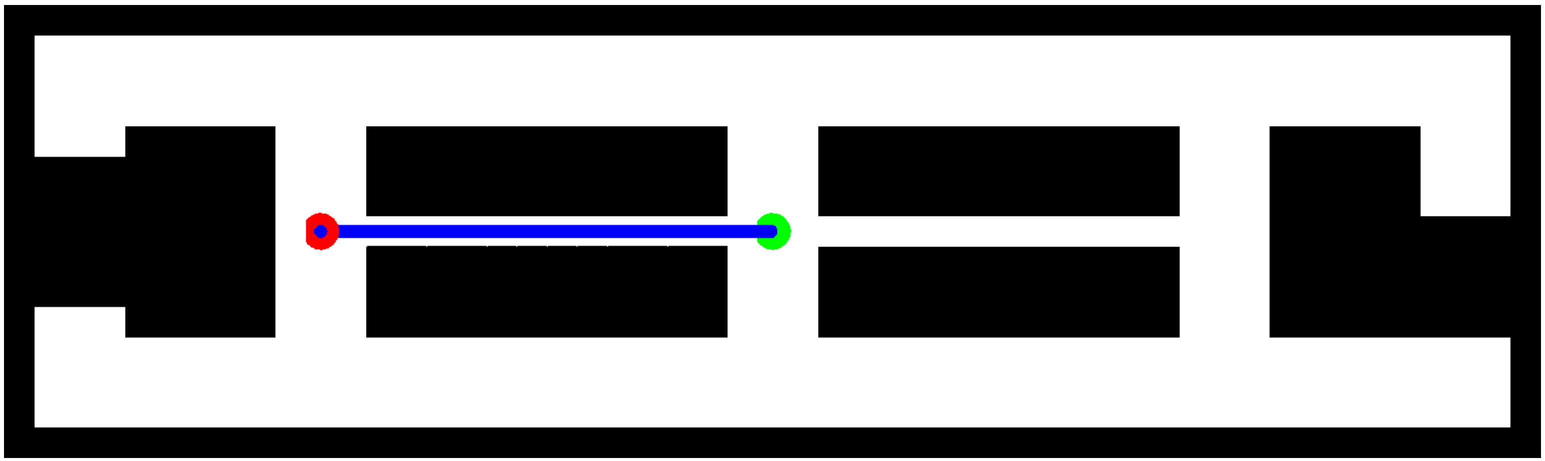
A Planning View



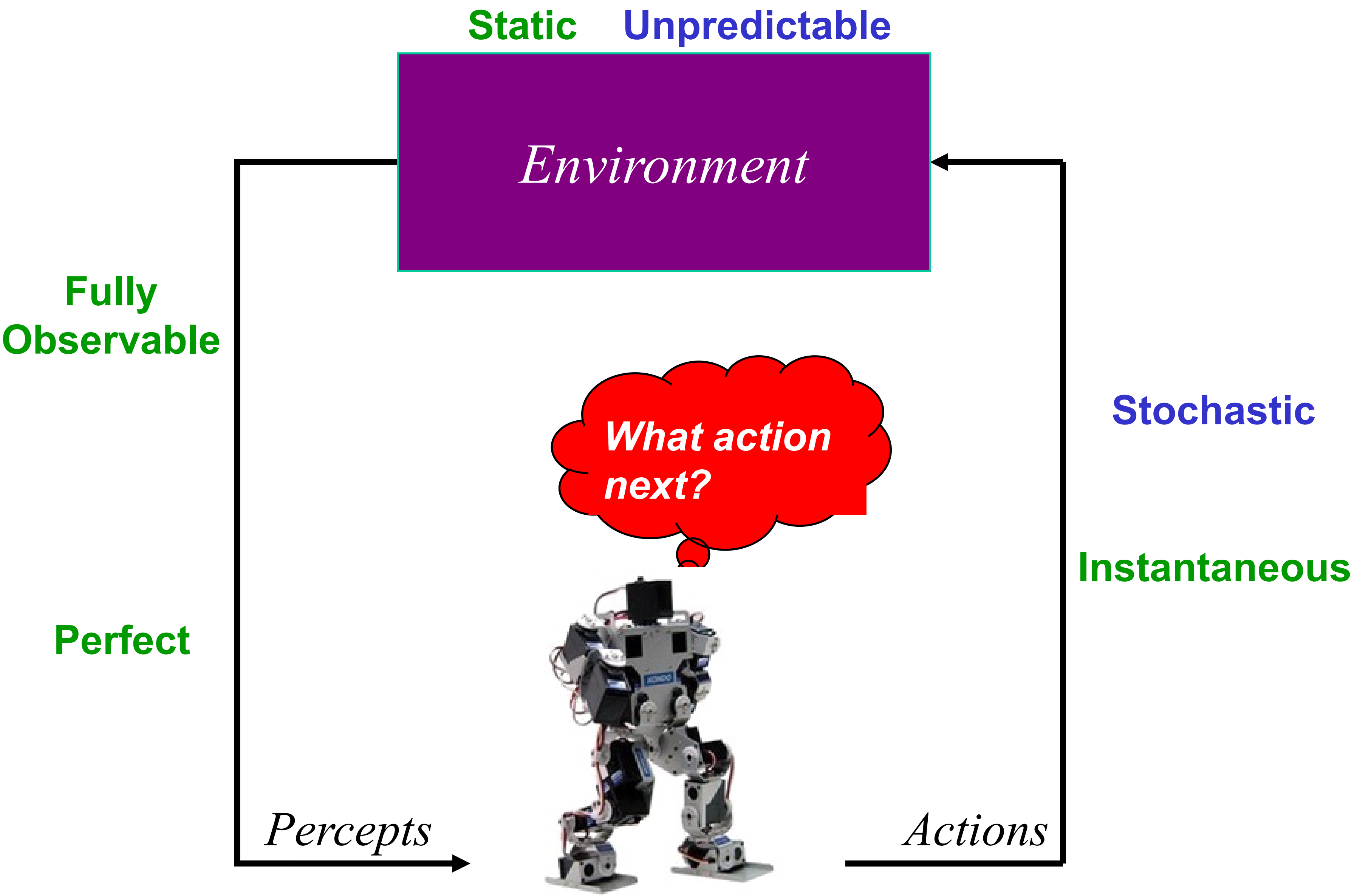
Classical Planning



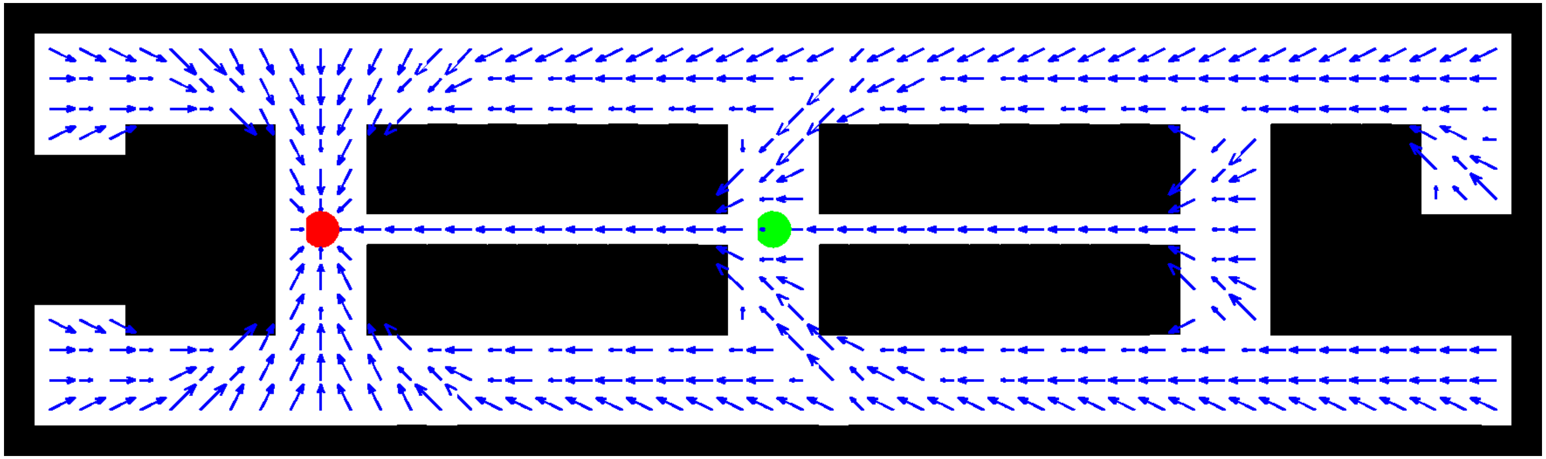
Deterministic, fully observable

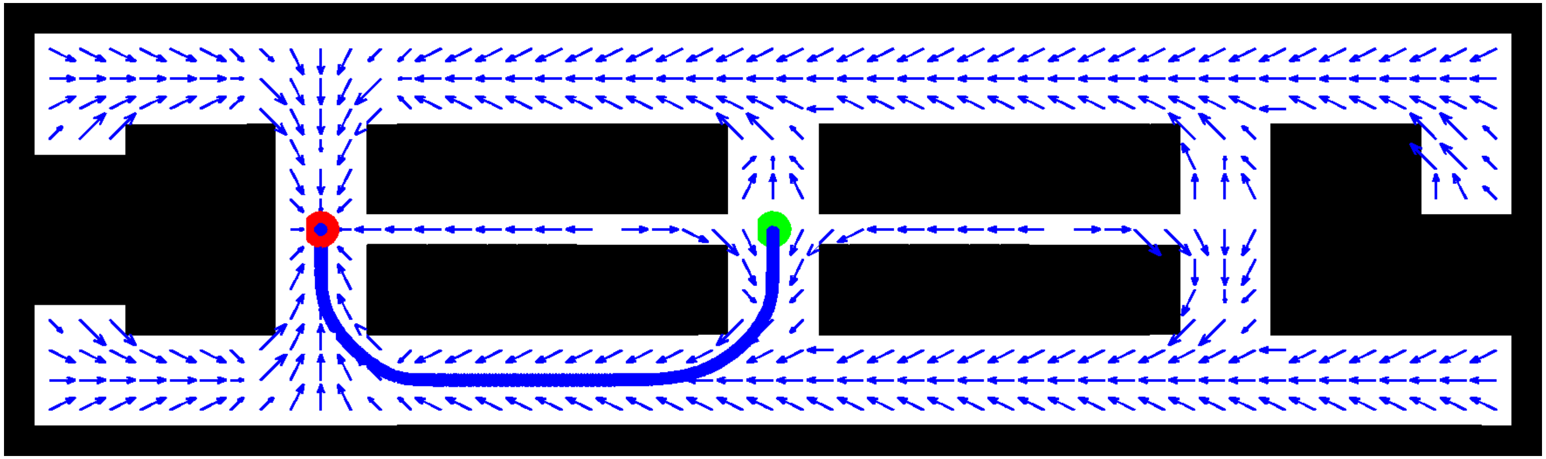


Stochastic Planning: MDPs



Stochastic, Fully Observable





Markov Decision Process (MDP)

- $\mathcal{S}$: A set of states
- $\mathcal{A}$: A set of actions
- $\mathcal{P}r(s'|s,a)$: transition model
- $\mathcal{C}(s,a,s')$: cost model
- $\mathcal{G}$: set of goals
- s_0: start state
- γ: discount factor
- $\mathcal{R}(s,a,s')$: reward model

factored

Factored MDP

absorbing/
non-absorbing

Objective of an MDP

- Find a policy $\pi: \mathcal{S} \rightarrow \mathcal{A}$

- which optimizes
  - minimizes $\begin{pmatrix} \text{discounted} \\ \text{or} \\ \text{undiscount.} \end{pmatrix}$ expected cost to reach a goal
  - maximizes expected reward
  - maximizes expected (reward-cost)

- given a ____ horizon
  - finite
  - infinite
  - indefinite

- assuming full observability

Role of Discount Factor (γ)

- Keep the total reward/total cost finite
  - useful for infinite horizon problems

- Intuition (economics):
  - Money today is worth more than money tomorrow.

- Total reward: $r_1 + \gamma r_2 + \gamma^2 r_3 + \ldots$
- Total cost: $c_1 + \gamma c_2 + \gamma^2 c_3 + \ldots$

Examples of MDPs

- Goal-directed, Indefinite Horizon, Cost Minimization MDP
  - $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}r, \mathcal{C}, \mathcal{G}, s_0 \rangle$
  - Most often studied in planning, graph theory communities
- Infinite Horizon, Discounted Reward Maximization MDP
  - $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}r, \mathcal{R}, \gamma \rangle$
  - Most often studied in machine learning, economics, operations research communities

most popular

- Goal-directed, Finite Horizon, Prob. Maximization MDP
  - $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}r, \mathcal{G}, s_0, T \rangle$
  - Also studied in planning community
- Oversubscription Planning: Non absorbing goals, Reward Max. MDP
  - $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}r, \mathcal{G}, \mathcal{R}, s_0 \rangle$
  - Relatively recent model

Bellman Equations for MDP_1

- <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{C}$, $\mathcal{G}$, s_0>
- Define J*(s) {optimal cost} as the minimum expected cost to reach a goal from this state.
- J* should satisfy the following equation:

$$
\begin{aligned}
J^*(s) &= 0 \ \ \textit{if} \ \ s \in \mathcal{G} \\
J^*(s) &= \min_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a) \left[\mathcal{C}(s,a,s') + J^*(s') \right]
\end{aligned}
$$

Bellman Equations for MDP_2

- <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{R}$, s_0, γ>
- Define **V*(s)** {optimal **value**} as the **maximum** expected **discounted reward** from this state.
- V* should satisfy the following equation:

$$V^*(s) \;=\; \max_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]$$

Bellman Equations for MDP_3

- $<\mathcal{S}, \mathcal{A}, \mathcal{P}r, \mathcal{G}, s_0, T>$
- Define **P*(s,t)** {optimal prob} as the maximum expected probability to reach a goal from this state starting **at tth timestep**.
- P* should satisfy the following equation:

$$
\begin{aligned}
P^*(s,t) &= 1 \;\; if \;\; s \in \mathcal{G} \\
P^*(s,T) &= 0 \;\; if \;\; s \notin \mathcal{G} \\
P^*(s,t) &= \max_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a) P^*(s', t+1)
\end{aligned}
$$

Bellman Backup (MDP_2)

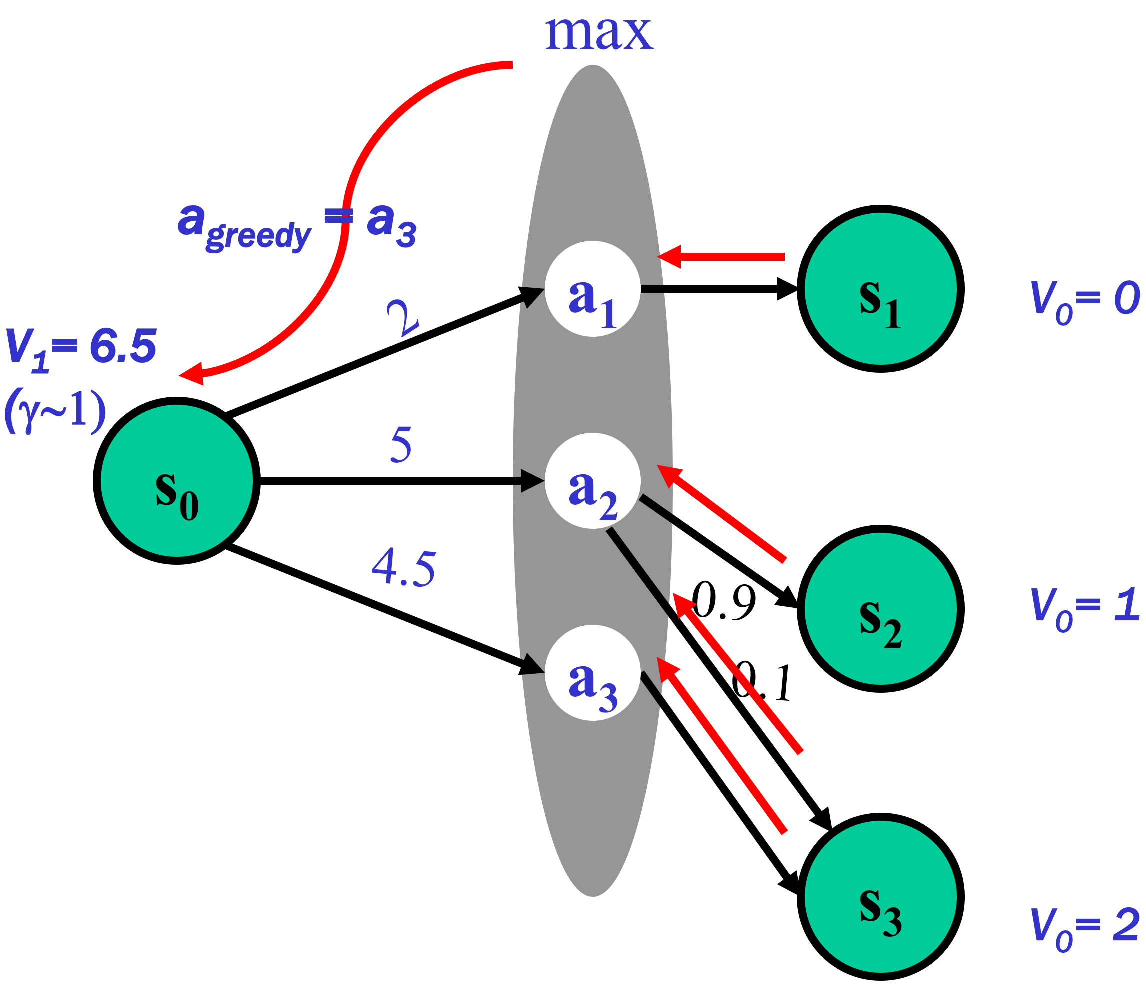
- Given an estimate of V* function (say V_n)
- Backup V_n function at state s
  - calculate a new estimate (V_{n+1}) :

$$Q_{n+1}(s,a) = \sum_{s' \in \mathcal{S}} Pr(s'|s,a) \left[\mathcal{R}(s,a,s') + \gamma V_n(s') \right]$$

$$V_{n+1}(s) = \max_{a \in Ap(s)} [Q_{n+1}(s,a)]$$

- $Q_{n+1}(s,a)$: value/cost of the strategy:
  - execute action a in s, execute π_n subsequently
  - $\pi_n = \text{argmax}_{a \in Ap(s)} Q_n(s,a)$

Bellman Backup



$Q_1(s,a_1) = 2 + 0\,\gamma$

$Q_1(s,a_2) = 5 + \gamma\, 0.9 \times 1 + \gamma\, 0.1 \times 2$

$Q_1(s,a_3) = 4.5 + 2\,\gamma$

Value iteration [Bellman'57]

- assign an arbitrary assignment of V_0 to each state.

- repeat
  - for all states s
    - compute $V_{n+1}(s)$ by Bellman backup at s.
- until $\max_s |V_{n+1}(s) - V_n(s)| < \varepsilon$

Iteration n+1

Residual(s)

ε-convergence

Comments

- Decision-theoretic Algorithm
- Dynamic Programming
- Fixed Point Computation
- Probabilistic version of Bellman-Ford Algorithm
  - for shortest path computation
  - MDP_1 : Stochastic Shortest Path Problem

- Time Complexity
  - one iteration: $O(|\mathcal{S}|^2|\mathcal{A}|)$
  - number of iterations: poly($|\mathcal{S}|$, $|\mathcal{A}|$, $1/(1-\gamma)$)
- Space Complexity: $O(|\mathcal{S}|)$
- Factored MDPs
  - exponential space, exponential time

Convergence Properties

- $V_n \rightarrow V^*$ in the limit as $n \rightarrow \infty$
- ε-convergence: V_n function is within ε of V^*
- Optimality: current policy is within $2\varepsilon\gamma/(1-\gamma)$ of optimal

- Monotonicity
  - $V_0 \leq_p V^* \Rightarrow V_n \leq_p V^*$ (V_n monotonic from below)
  - $V_0 \geq_p V^* \Rightarrow V_n \geq_p V^*$ (V_n monotonic from above)
  - otherwise V_n non-monotonic

Policy Computation

$$\pi^*(s) = \operatorname*{argmax}_{a \in Ap(s)} Q^*(s,a)$$
$$= \operatorname*{argmax}_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a) \left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]$$

Policy Evaluation

$$V_\pi(s) = \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,\pi(s)) \left[\mathcal{R}(s,\pi(s),s') + \gamma V_\pi(s')\right]$$

A system of linear equations in $|\mathcal{S}|$ variables.

Changing the Search Space

- Value Iteration
  - Search in value space
  - Compute the resulting policy

- Policy Iteration
  - Search in policy space
  - Compute the resulting value

Policy iteration [Howard'60]

- assign an arbitrary assignment of π_0 to each state.
- repeat
  - Policy Evaluation: compute V_{n+1}: the evaluation of π_n
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmax}_{a\in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

costly: $O(n^3)$

Modified Policy Iteration → approximate by value iteration using fixed policy

Advantage

- searching in a finite (policy) space as opposed to uncountably infinite (value) space ⇒ convergence faster.
- all other properties follow!

Modified Policy iteration

- assign an arbitrary assignment of π_0 to each state.

- repeat
  - Policy Evaluation: compute V_{n+1} the *approx.* evaluation of π_n
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\mathrm{argmax}_{a\in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

Advantage

- probably the most competitive synchronous dynamic programming algorithm.

Asynchronous Value Iteration

- States may be backed up in any order
  - instead of an iteration by iteration
- As long as all states backed up infinitely often
  - Asynchronous Value Iteration converges to optimal

Asynch VI: Prioritized Sweeping

- Why backup a state if values of successors same?
- Prefer backing a state
  - whose successors had most change

- Priority Queue of (state, expected change in value)
- Backup in the order of priority
- After backing a state update priority queue
  - for all predecessors

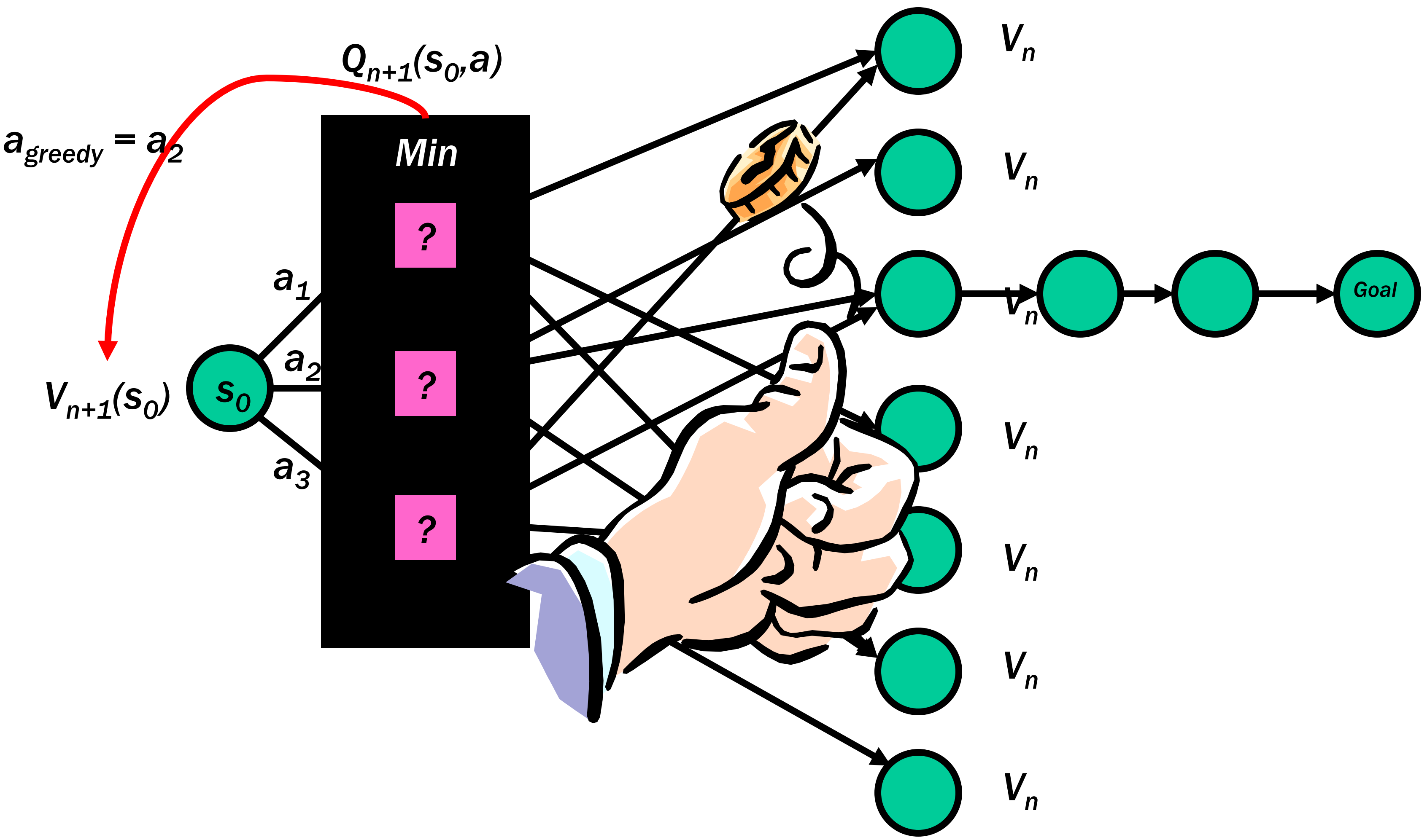
Asynch VI: Real Time Dynamic Programming

[Barto, Bradtke, Singh'95]

- Trial: simulate greedy policy starting from start state;
  perform Bellman backup on visited states

- RTDP: repeat Trials until value function converges

RTDP Trial



Comments

- Properties
  - if all states are visited infinitely often then $V_n \rightarrow V^*$
- Advantages
  - Anytime: more probable states explored quickly
- Disadvantages
  - complete convergence can be slow!

Reinforcement Learning

Reinforcement Learning

- Still have an MDP
  - Still looking for policy π
- New twist: don't know $\mathcal{P}$r and/or $\mathcal{R}$
  - i.e. don't know which states are good
  - and what actions do
- Must actually try out actions to learn

Model based methods

- Visit different states, perform different actions
- Estimate $\mathcal{P}r$ and $\mathcal{R}$

- Once model built, do planning using V.I. or other methods

- Con: require _huge_ amounts of data

Model free methods

- Directly learn Q*(s,a) values

$$Q^*(s,a) = \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]$$

$$Q^*(s,a) = \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma max_{a'} Q^*(s',a')\right]$$

- sample = $\mathcal{R}$(s,a,s') + $\gamma \max_{a'} Q_n(s',a')$
- Nudge the old estimate towards the new sample
- $Q_{n+1}(s,a) \leftarrow (1-\alpha)Q_n(s,a) + \alpha[\text{sample}]$

Properties

- Converges to optimal if
  - If you explore enough
  - If you make learning rate (α) small enough
  - But not decrease it too quickly

  - $\sum_i \alpha(s,a,i) = \infty$
  - $\sum_i \alpha^2(s,a,i) < \infty$

  where i is the number of visits to (s,a)

Model based vs. Model Free RL

- Model based
  - estimate $O(|\mathcal{S}|^2|\mathcal{A}|)$ parameters
  - requires relatively larger data for learning
  - can make use of background knowledge easily

- Model free
  - estimate $O(|\mathcal{S}||\mathcal{A}|)$ parameters
  - requires relatively less data for learning

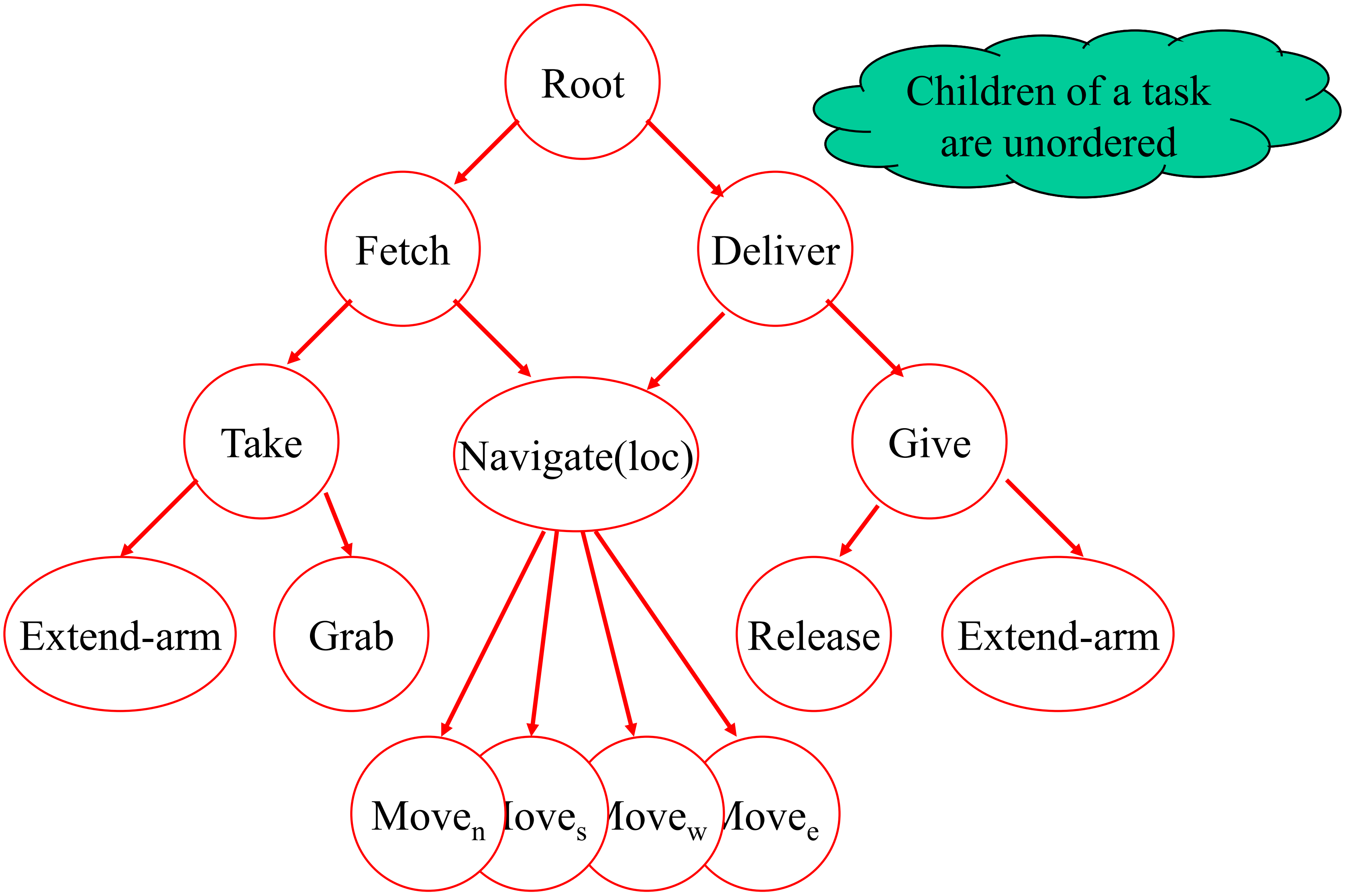
Exploration vs. Exploitation

- Exploration: choose actions that visit new states in order to obtain more data for better learning.
- Exploitation: choose actions that maximize the reward given current learnt model.
- ε-greedy
  - Each time step flip a coin
  - With prob ε, take an action randomly
  - With prob 1-ε take the current greedy action
- Lower ε over time
  - increase exploitation as more learning has happened

Q-learning

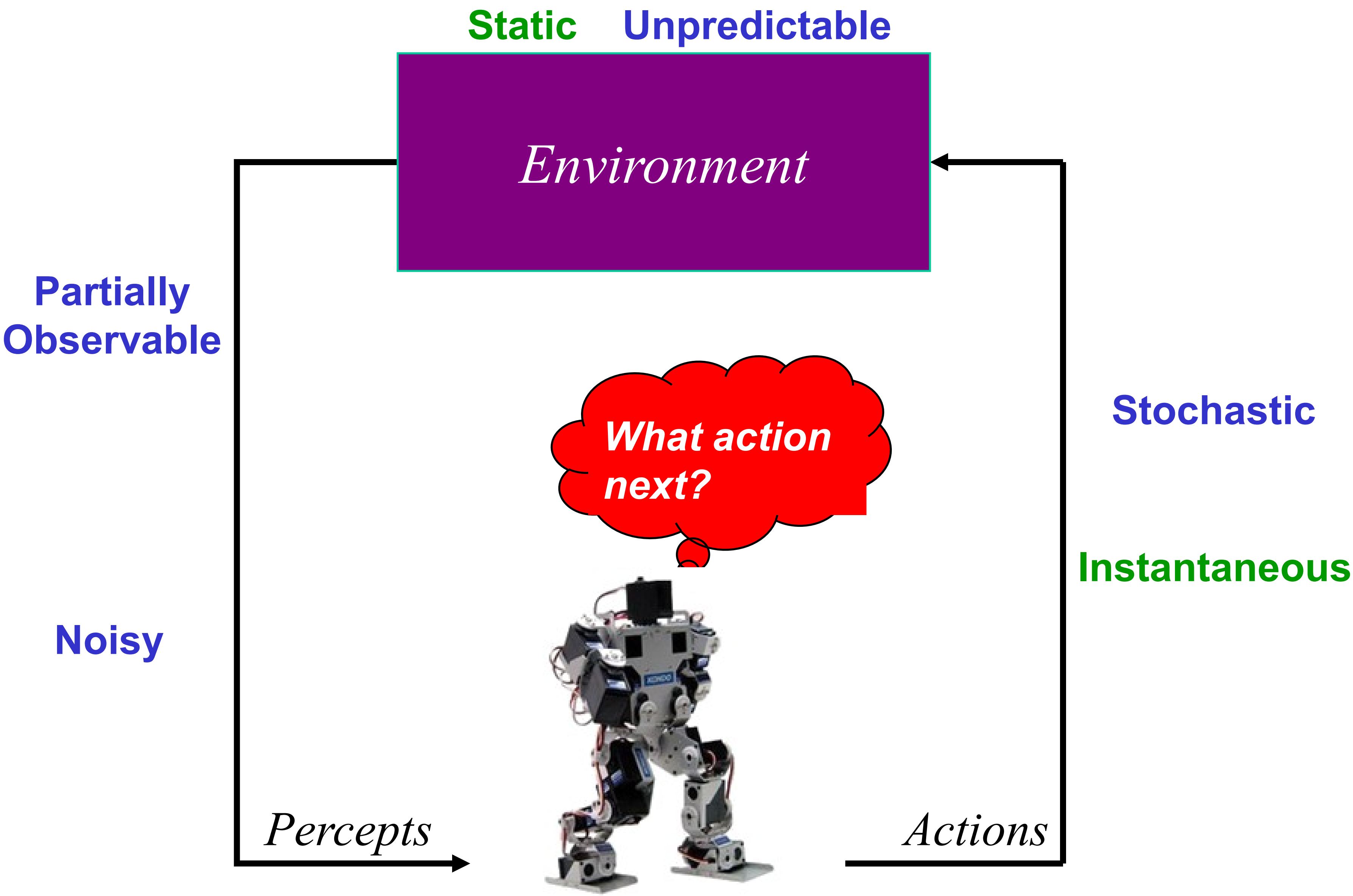
- Problems
  - Too many states to visit during learning
  - Q(s,a) is still a BIG table
- We want to generalize from small set of training examples
- Techniques
  - Value function approximators
  - Policy approximators
  - Hierarchical Reinforcement Learning

Task Hierarchy: MAXQ Decomposition [Dietterich'00]

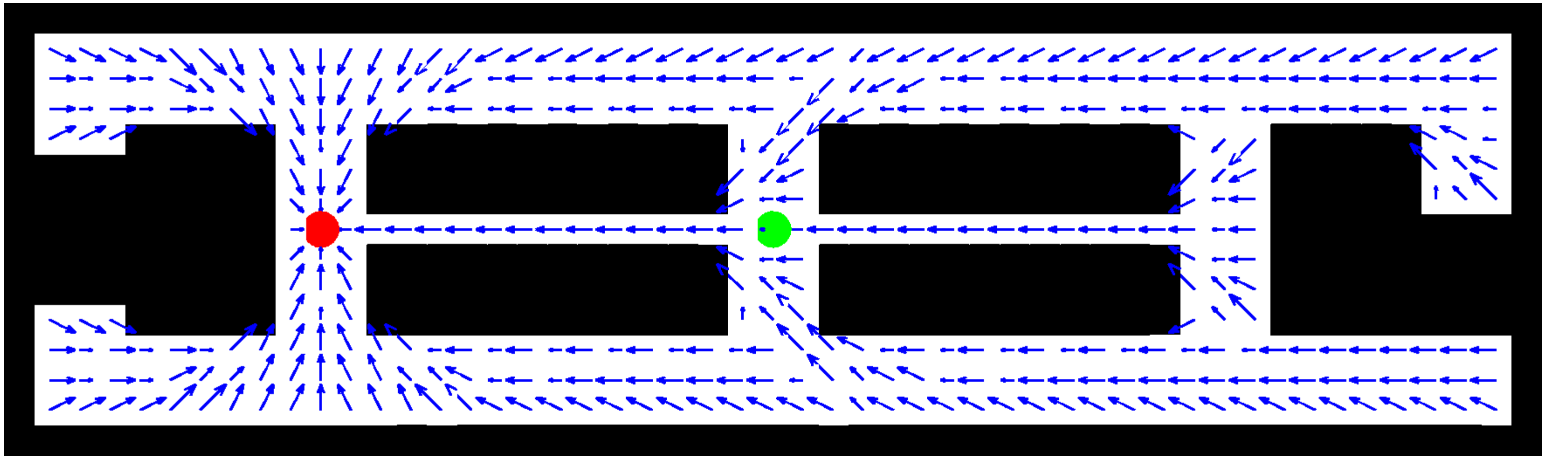


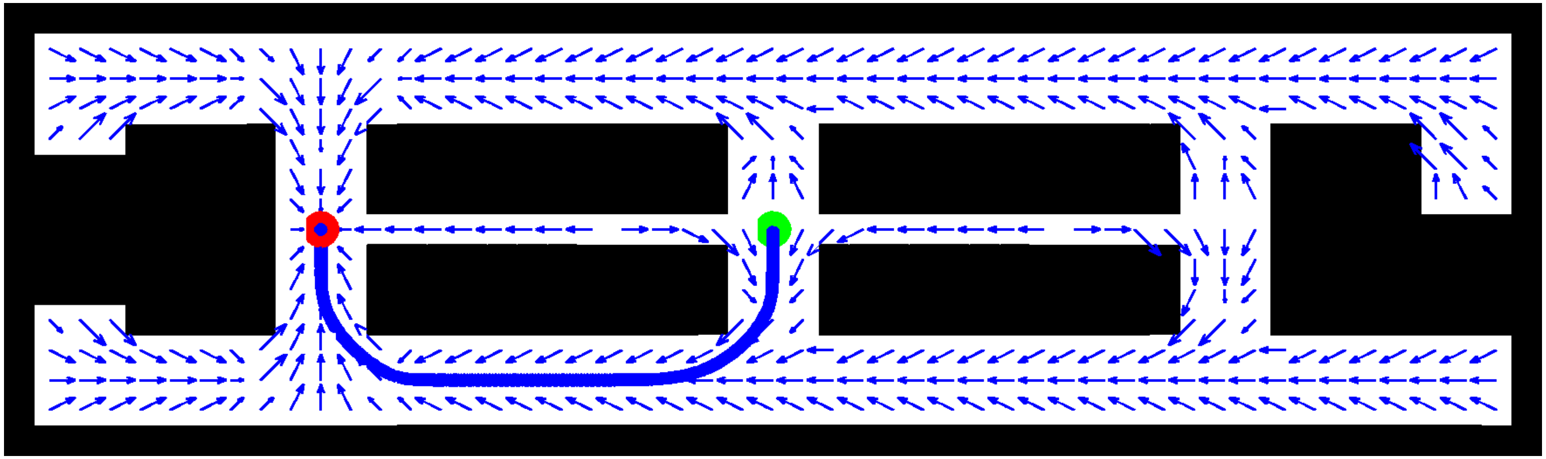
Partially Observable Markov Decision Processes

Partially Observable MDPs

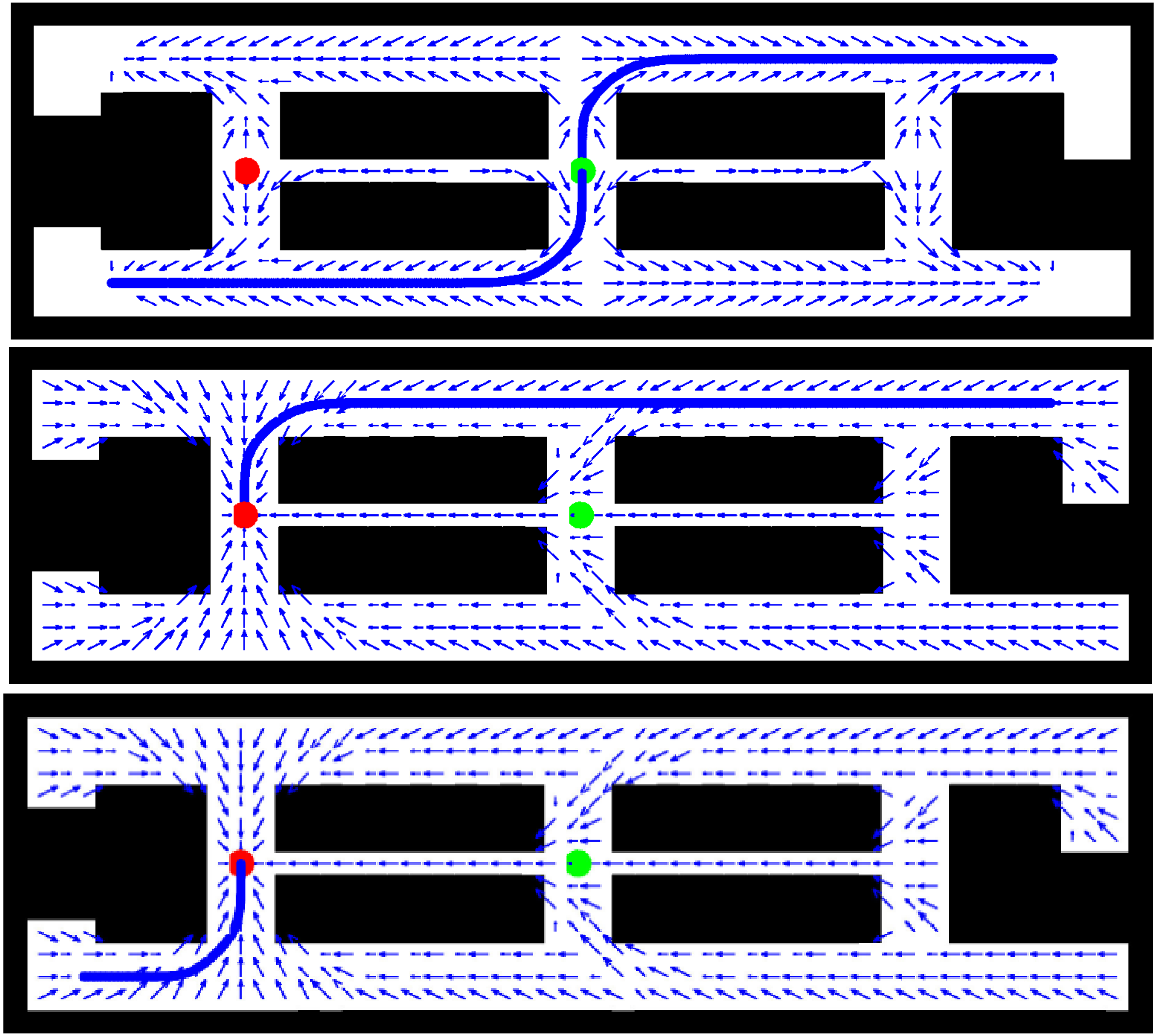


Stochastic, Fully Observable





Stochastic, Partially Observable



POMDPs

- In POMDPs we apply the very same idea as in MDPs.

- Since the state is not observable,

  the agent has to make its decisions based on the belief state which is a posterior distribution over states.

- Let b be the belief of the agent about the current state
- POMDPs compute a value function over belief space:

$$V_T(b) \quad = \quad \max_a \left[r(b,a) + \gamma \int V_{T-1}(b')p(b' \mid b,a)\, db' \right]$$

POMDPs

- Each belief is a probability distribution,
  - value fn is a function of an entire probability distribution.
- Problematic, since probability distributions are continuous.
- Also, we have to deal with huge complexity of belief spaces.

- For finite worlds with finite state, action, and observation spaces and finite horizons,
  - we can represent the value functions by piecewise linear functions.

Applications

- Robotic control
  - helicopter maneuvering, autonomous vehicles
  - Mars rover - path planning, oversubscription planning
  - elevator planning
- Game playing - backgammon, tetris, checkers
- Neuroscience
- Computational Finance, Sequential Auctions
- Assisting elderly in simple tasks
- Spoken dialog management
- Communication Networks – switching, routing, flow control
- War planning, evacuation planning