Review

\[ e ::= \lambda x. e \mid x \mid e\ e \mid c \]
\[ v ::= \lambda x. e \mid c \]
\[ \tau ::= \text{int} \mid \tau \rightarrow \tau \]
\[ \Gamma ::= \cdot \mid \Gamma, x : \tau \]

\[
\begin{align*}
(\lambda x. e)\ v & \rightarrow e[v/x] \\
\end{align*}
\]

\[
\begin{align*}
e_1 \rightarrow e'_1 \\
e_2 \rightarrow e'_2 \\
e_1\ e_2 & \rightarrow e'_1\ e_2 \\
v\ e_2 & \rightarrow v\ e'_2
\end{align*}
\]

\[
e[e'/x]: \text{capture-avoiding substitution of } e' \text{ for } x \text{ in } e
\]

\[
\begin{align*}
\Gamma \vdash c : \text{int} \\
\Gamma \vdash x : \Gamma(x) \\
\Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 \\
\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \\
\Gamma \vdash e_2 : \tau_2 \\
\Gamma \vdash e_1\ e_2 : \tau_1
\end{align*}
\]

Preservation: If \( \cdot \vdash e : \tau \) and \( e \rightarrow e' \), then \( \cdot \vdash e' : \tau \).
Progress: If \( \cdot \vdash e : \tau \), then \( e \) is a value or \( \exists e' \) such that \( e \rightarrow e' \).
Adding Stuff

Time to use STLC as a foundation for understanding other common language constructs

We will add things via a *principled methodology* thanks to a *proper education*

- Extend the syntax
- Extend the operational semantics
  - Derived forms (syntactic sugar), or
  - Direct semantics
- Extend the type system
- Extend soundness proof (new stuck states, proof cases)

In fact, extensions that add new types have even more structure
Let bindings (CBV)

\[ e ::= \cdots \mid \text{let } x = e_1 \text{ in } e_2 \]

\[
\begin{align*}
    & e_1 \rightarrow e'_1 \\
    \Rightarrow & \quad \text{let } x = e_1 \text{ in } e_2 \rightarrow \text{let } x = e'_1 \text{ in } e_2 \\
    & \text{let } x = v \text{ in } e \rightarrow e[v/x]
\end{align*}
\]

\[
\begin{align*}
    & \Gamma \vdash e_1 : \tau' \\
    & \Gamma, x : \tau' \vdash e_2 : \tau \\
    \Rightarrow & \quad \Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau
\end{align*}
\]

(Also need to extend definition of substitution...)

Progress: If \( e \) is a let, 1 of the 2 new rules apply (using induction)

Preservation: Uses Substitution Lemma

Substitution Lemma: Uses Weakening and Exchange
Derived forms

let seems just like \( \lambda \), so can make it a derived form

- **let** \( x = e_1 \text{ in } e_2 \) “a macro” / “desugars to” \((\lambda x. e_2) \ e_1\)

- A “derived form”

(Harder if \( \lambda \) needs explicit type)

Or just define the semantics to replace let with \( \lambda \):

\[
\text{let } x = e_1 \text{ in } e_2 \rightarrow (\lambda x. e_2) \ e_1
\]

These 3 semantics are *different* in the state-sequence sense
\((e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_n)\)

- But (totally) *equivalent* and you could prove it (not hard)

Note: ML type-checks let and \( \lambda \) differently (later topic)
Note: Don’t desugar early if it hurts error messages!
Booleans and Conditionals

\[
e ::= \cdots \mid \text{true} \mid \text{false} \mid \text{if } e_1 e_2 e_3
\]

\[
\tau ::= \cdots \mid \text{bool}
\]

\[
\frac{e_1 \rightarrow e_1'}{\text{if } e_1 e_2 e_3 \rightarrow \text{if } e_1' e_2 e_3}
\]

\[
\frac{\text{if true } e_2 e_3 \rightarrow e_2}{\text{if false } e_2 e_3 \rightarrow e_3}
\]

\[
\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if } e_1 e_2 e_3 : \tau}
\]

\[
\frac{\Gamma \vdash \text{true} : \text{bool}}{\Gamma \vdash \text{false} : \text{bool}}
\]

Also extend definition of substitution (will stop writing that)...

Notes: CBN, new Canonical Forms case, all lemma cases easy
Pairs (CBV, left-right)

\[
e ::= \cdots \mid (e,e) \mid e.1 \mid e.2
\]

\[
v ::= \cdots \mid (v,v)
\]

\[
\tau ::= \cdots \mid \tau \ast \tau
\]

\[
\begin{align*}
e_1 &\rightarrow e'_1 \\
(e_1, e_2) &\rightarrow (e'_1, e_2)
\end{align*}
\]

\[
\begin{align*}
e &\rightarrow e' \\
e.1 &\rightarrow e'.1
\end{align*}
\]

\[
\begin{align*}
e_2 &\rightarrow e'_2 \\
(v_1, e_2) &\rightarrow (v_1, e'_2)
\end{align*}
\]

\[
\begin{align*}
e &\rightarrow e' \\
e.2 &\rightarrow e'.2
\end{align*}
\]

\[
\begin{align*}
(v_1, v_2).1 &\rightarrow v_1 \\
(v_1, v_2).2 &\rightarrow v_2
\end{align*}
\]

Small-step can be a pain

- Large-step needs only 3 rules
- Will learn more concise notation later (evaluation contexts)
Pairs continued

\[ \Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2 \]
\[ \Gamma \vdash (e_1, e_2) : \tau_1 \ast \tau_2 \]

\[ \Gamma \vdash e : \tau_1 \ast \tau_2 \]
\[ \Gamma \vdash e.1 : \tau_1 \]
\[ \Gamma \vdash e.2 : \tau_2 \]

Canonical Forms: If \( \cdot \vdash v : \tau_1 \ast \tau_2 \), then \( v \) has the form \((v_1, v_2)\)

Progress: New cases using Canonical Forms are \( v.1 \) and \( v.2 \)

Preservation: For primitive reductions, inversion gives the result \textit{directly}
Records are like $n$-ary tuples except with named fields

- Field names are *not* variables; they do *not* $\alpha$-convert

\[ e ::= \cdots \mid \{l_1 = e_1; \ldots; l_n = e_n\} \mid e.l \]

\[ v ::= \cdots \mid \{l_1 = v_1; \ldots; l_n = v_n\} \]

\[ \tau ::= \cdots \mid \{l_1 : \tau_1; \ldots; l_n : \tau_n\} \]

\[ e_i \rightarrow e'_i \]
\[
\{l_1 = v_1, \ldots, l_{i-1} = v_{i-1}, l_i = e_i, \ldots, l_n = e_n\} \\
\rightarrow \{l_1 = v_1, \ldots, l_{i-1} = v_{i-1}, l_i = e'_i, \ldots, l_n = e_n\}
\]

\[ 1 \leq i \leq n \]
\[
\{l_1 = v_1, \ldots, l_n = v_n\}.l_i \rightarrow v_i
\]
Records continued

Should we be allowed to reorder fields?

- ⊢ \{l_1 = 42; l_2 = true\} : \{l_2 : bool; l_1 : int\} ??
- Really a question about, “when are two types equal?”

Nothing wrong with this from a type-safety perspective, yet many languages disallow it

- Reasons: Implementation efficiency, type inference

Return to this topic when we study subtyping
Sums

What about ML-style datatypes:

```plaintext
type t = A | B of int | C of int * t
```

1. Tagged variants (i.e., discriminated unions)

2. Recursive types

3. Type constructors (e.g., type 'a mylist = ...)

4. Named types

For now, just model (1) with (anonymous) sum types

- (2) is in a later lecture, (3) is straightforward, and (4) we’ll discuss informally
Sums syntax and overview

\[ e ::= \cdots | A(e) | B(e) | \text{match } e \text{ with } A x. \ e | B x. \ e \]
\[ v ::= \cdots | A(v) | B(v) \]
\[ \tau ::= \cdots | \tau_1 + \tau_2 \]

➤ Only two constructors: \( A \) and \( B \)

➤ All values of any sum type built from these constructors

➤ So \( A(e) \) can have any sum type allowed by \( e \)'s type

➤ No need to declare sum types in advance

➤ Like functions, will “guess the type” in our rules
Sums operational semantics

\[
\text{match } A(v) \text{ with } Ax. \; e_1 \mid By. \; e_2 \rightarrow e_1[v/x]
\]

\[
\text{match } B(v) \text{ with } Ax. \; e_1 \mid By. \; e_2 \rightarrow e_2[v/y]
\]

\[
e \rightarrow e' \quad \frac{A(e) \rightarrow A(e')} \quad B(e) \rightarrow B(e')
\]

\[
e \rightarrow e' \rightarrow e' \rightarrow e' \rightarrow e' \rightarrow e'
\]

\[
\text{match } e \text{ with } Ax. \; e_1 \mid By. \; e_2 \rightarrow \text{match } e' \text{ with } Ax. \; e_1 \mid By. \; e_2
\]

\text{match} \text{ has binding occurrences, just like pattern-matching}

(Definition of substitution must avoid capture, just like functions)
What is going on

Feel free to think about *tagged values* in your head:

- A tagged value is a pair of:
  - A tag A or B (or 0 or 1 if you prefer)
  - The (underlying) value

- A match:
  - Checks the tag
  - Binds the variable to the (underlying) value

This much is just like OCaml and related to homework 2
Sums Typing Rules

Inference version (not trivial to infer; can require annotations)

\[
\begin{align*}
\Gamma \vdash e : \tau_1 & \quad \Gamma \vdash e : \tau_2 \\
\Gamma \vdash A(e) : \tau_1 + \tau_2 & \quad \Gamma \vdash B(e) : \tau_1 + \tau_2 \\
\Gamma \vdash e : \tau_1 + \tau_2 & \quad \Gamma, x:\tau_1 \vdash e_1 : \tau \\
\Gamma, y:\tau_2 \vdash e_2 : \tau & \\
\Gamma \vdash \text{match } e \text{ with } Ax. e_1 | By. e_2 : \tau
\end{align*}
\]

Key ideas:

- For constructor-uses, “other side can be anything”
- For \texttt{match}, both sides need same type
  - Don’t know which branch will be taken, just like an \texttt{if}.
  - In fact, can drop explicit bools and encode with sums:
    E.g., \texttt{bool = int + int}, \texttt{true = A(0)}, \texttt{false = B(0)}
Sums Type Safety

Canonical Forms: If $\vdash v : \tau_1 + \tau_2$, then there exists a $v_1$ such that either $v$ is $A(v_1)$ and $\vdash v_1 : \tau_1$ or $v$ is $B(v_1)$ and $\vdash v_1 : \tau_2$

- Progress for `match v with A x. e_1 | B y. e_2` follows, as usual, from Canonical Forms
- Preservation for `match v with A x. e_1 | B y. e_2` follows from the type of the underlying value and the Substitution Lemma
- The Substitution Lemma has new “hard” cases because we have new binding occurrences
- But that’s all there is to it (plus lots of induction)
What are sums for?

- Pairs, structs, records, aggregates are fundamental data-builders

- Sums are just as fundamental: “this or that not both”

- You have seen how OCaml does sums (datatypes)

- Worth showing how C and Java do the same thing
  - A primitive in one language is an idiom in another
Sums in C

type t = A of t1 | B of t2 | C of t3
match e with A x -> ...

One way in C:

```c
struct t {
  enum {A, B, C} tag;
  union {t1 a; t2 b; t3 c;} data;
};
... switch(e->tag){ case A: t1 x=e->data.a; ...
```

- No static checking that tag is obeyed
- As fat as the fattest variant (avoidable with casts)
  - Mutation costs us again!
Sums in Java

type t = A of t1 | B of t2 | C of t3
match e with A x -> ...

One way in Java (t4 is the match-expression’s type):

```java
abstract class t {abstract t4 m();}
class A extends t { t1 x; t4 m(){...}}
class B extends t { t2 x; t4 m(){...}}
class C extends t { t3 x; t4 m(){...}}
... e.m() ...
```

- A new method in t and subclasses for each match expression
- Supports extensibility via new variants (subclasses) instead of extensibility via new operations (**match** expressions)
Pairs vs. Sums

You need both in your language

- With only pairs, you clumsily use dummy values, waste space, and rely on unchecked tagging conventions

- Example: replace `int + (int → int)` with `int * (int * (int → int))`

Pairs and sums are “logical duals” (more on that later)

- To make a $\tau_1 \ast \tau_2$ you need a $\tau_1$ and a $\tau_2$
- To make a $\tau_1 + \tau_2$ you need a $\tau_1$ or a $\tau_2$
- Given a $\tau_1 \ast \tau_2$, you can get a $\tau_1$ or a $\tau_2$ (or both; your “choice”)
- Given a $\tau_1 + \tau_2$, you must be prepared for either a $\tau_1$ or $\tau_2$ (the value’s “choice”)
Base Types and Primitives, in general

What about floats, strings, ...? Could add them all or do something more general...

Parameterize our language/semantics by a collection of base types \((b_1, \ldots, b_n)\) and primitives \((p_1 : \tau_1, \ldots, p_n : \tau_n)\). Examples:

- `concat : string → string → string`
- `toInt : float → int`
- “hello” : string

For each primitive, assume if applied to values of the right types it produces a value of the right type.

Together the types and assumed steps tell us how to type-check and evaluate \(p_i \; v_1 \ldots v_n\) where \(p_i\) is a primitive.

We can prove soundness once and for all given the assumptions.
Recursion

We won’t prove it, but every extension so far preserves termination

A Turing-complete language needs some sort of loop, but our lambda-calculus encoding won’t type-check, nor will any encoding of equal expressive power

- So instead add an explicit construct for recursion
- You might be thinking \textbf{let rec }\textit{f x} = \textit{e}, but we will do something more concise and general but less intuitive

\[
\text{fix } e \rightarrow e
\]
\[
\text{fix } \lambda x. e \rightarrow e [\text{fix } \lambda x. e/x]
\]

No new values and no new types
Recursion

We won’t prove it, but every extension so far preserves termination.

A Turing-complete language needs some sort of loop, but our lambda-calculus encoding won’t type-check, nor will any encoding of equal expressive power.

- So instead add an explicit construct for recursion.
- You might be thinking `let rec f x = e`, but we will do something more concise and general but less intuitive.

\[
e ::= \cdots | \text{fix } e
\]

\[
e \rightarrow e' \\
\text{fix } e \rightarrow \text{fix } e' \\
\text{fix } \lambda x. e \rightarrow e[\text{fix } \lambda x. e/x]
\]

No new values and no new types.
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

- Not shown: fix and tuples can also encode mutual recursion

Example:

$$(\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1))))\ 5$$
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

- Not shown: fix and tuples can also encode mutual recursion

Example:

$$(\text{fix } \lambda f. \lambda n. \text{if } (n<1) \ 1 \ (n \ast (f(n - 1)))) \ 5$$

$$\rightarrow$$

$$(\lambda n. \text{if } (n<1) \ 1 \ (n \ast ((\text{fix } \lambda f. \lambda n. \text{if } (n<1) \ 1 \ (n \ast (f(n - 1))))(n - 1)))) \ 5$$
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

- Not shown: fix and tuples can also encode mutual recursion

Example:

\[
(fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \times (f(n-1)))) 5
\]

\[
\rightarrow
\]

\[
(\lambda n. \text{if } (n<1) 1 (n \times ((fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \times (f(n-1))))(n - 1)))) 5
\]

\[
\rightarrow
\]

\[
\text{if } (5<1) 1 (5 \times ((fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \times (f(n-1))))(5 - 1))
\]
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

- Not shown: fix and tuples can also encode mutual recursion

Example:

\[
(fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \ast (f(n - 1)))) \ 5
\]

→

\[
(\lambda n. \text{if } (n<1) 1 (n \ast ((fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \ast (f(n - 1))))(n - 1)))) \ 5
\]

→

\[
\text{if } (5<1) 1 (5 \ast ((fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \ast (f(n - 1))))(5 - 1))
\]

→

\[
5 \ast ((fix \lambda f. \lambda n. \text{if } (n<1) 1 (n \ast (f(n - 1))))(5 - 1))
\]
Using fix

To use \texttt{fix} like \texttt{let rec}, just pass it a two-argument function where the first argument is for recursion

- Not shown: \texttt{fix} and tuples can also encode mutual recursion

Example:
\[
(\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1)))) 5
\]
\[
\rightarrow
\]
\[
(\lambda n. \text{if } (n < 1) 1 (n \ast ((\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1))))(n - 1)))) 5
\]
\[
\rightarrow
\]
\[
\text{if } (5 < 1) 1 (5 \ast ((\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1))))(5 - 1))
\]
\[
\rightarrow^2
\]
\[
5 \ast ((\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1))))(5 - 1))
\]
\[
\rightarrow^2
\]
\[
5 \ast ((\lambda n. \text{if } (n < 1) 1 (n \ast ((\text{fix } \lambda f. \lambda n. \text{if } (n < 1) 1 (n \ast (f(n - 1))))(n - 1))))(5 - 1)) 4)
\]
\[
\rightarrow
\]
\[
\ldots
\]
Why called fix?

In math, a fix-point of a function $g$ is an $x$ such that $g(x) = x$

- This makes sense only if $g$ has type $\tau \rightarrow \tau$ for some $\tau$
- A particular $g$ could have have 0, 1, 39, or infinity fix-points
- Examples for functions of type $\text{int} \rightarrow \text{int}$:
  - $\lambda x. x + 1$ has no fix-points
  - $\lambda x. x \times 0$ has one fix-point
  - $\lambda x. \text{absolute\_value}(x)$ has an infinite number of fix-points
  - $\lambda x. \text{if } (x < 10 \&\& x > 0) \text{ then } x \text{ else } 0$ has 10 fix-points
Higher types

At higher types like \((\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})\), the notion of fix-point is exactly the same (but harder to think about)

- For what inputs \(f\) of type \(\text{int} \rightarrow \text{int}\) is \(g(f) = f\)

Examples:

- \(\lambda f. \lambda x. (f x) + 1\) has no fix-points

- \(\lambda f. \lambda x. (f x) \ast 0\) (or just \(\lambda f. \lambda x. 0\)) has 1 fix-point
  - The function that always returns 0
  - In math, there is exactly one such function (cf. equivalence)

- \(\lambda f. \lambda x. \text{absolute_value}(f x)\) has an infinite number of fix-points: Any function that never returns a negative result
Back to factorial

Now, what are the fix-points of
\( \lambda f. \lambda x. \text{if } (x < 1) 1 \ (x \ast (f(x - 1))) \)?

It turns out there is exactly one (in math): the factorial function!

And \textbf{fix} \( \lambda f. \lambda x. \text{if } (x < 1) 1 \ (x \ast (f(x - 1))) \) behaves just like the factorial function

- That is, it behaves just like the fix-point of
  \( \lambda f. \lambda x. \text{if } (x < 1) 1 \ (x \ast (f(x - 1))) \)

- In general, \textbf{fix} takes a function-taking-function and returns its fix-point

(This isn't necessarily important, but it explains the terminology and shows that programming is deeply connected to mathematics)
Typing \texttt{fix}

\[
\Gamma \vdash e : \tau \rightarrow \tau \\
\Gamma \vdash \texttt{fix} \ e : \tau
\]

Math explanation: If \( e \) is a function from \( \tau \) to \( \tau \), then \texttt{fix} \( e \), the fixed-point of \( e \), is some \( \tau \) with the fixed-point property

▶ So it’s something with type \( \tau \)

Operational explanation: \texttt{fix} \( \lambda x. \ e' \) becomes \( e'[\texttt{fix} \ \lambda x. \ e'/x] \)

▶ The substitution means \( x \) and \texttt{fix} \( \lambda x. \ e' \) need the same type

▶ The result means \( e' \) and \texttt{fix} \( \lambda x. \ e' \) need the same type

Note: The \( \tau \) in the typing rule is usually insantiated with a function type

▶ e.g., \( \tau_1 \rightarrow \tau_2 \), so \( e \) has type \((\tau_1 \rightarrow \tau_2) \rightarrow (\tau_1 \rightarrow \tau_2)\)

Note: Proving soundness is straightforward!
General approach

We added let, booleans, pairs, records, sums, and fix

- **let** was syntactic sugar
- **fix** made us Turing-complete by “baking in” self-application
- The others *added types*

Whenever we add a new form of type \( \tau \) there are:

- Introduction forms (ways to make values of type \( \tau \))
- Elimination forms (ways to use values of type \( \tau \))

What are these forms for functions? Pairs? Sums?

When you add a new type, think “what are the intro and elim forms”? 
Anonymity

We added many forms of types, all unnamed a.k.a. structural. Many real PLs have (all or mostly) named types:

- Java, C, C++: all record types (or similar) have names
  - Omitting them just means compiler makes up a name
- OCaml sum types and record types have names

A never-ending debate:

- Structural types allow more code reuse: good
- Named types allow less code reuse: good
- Structural types allow generic type-based code: good
- Named types let type-based code distinguish names: good

The theory is often easier and simpler with structural types
Termination

Surprising fact: If $\vdash e : \tau$ in STLC with all our additions except $\text{fix}$, then there exists a $v$ such that $e \rightarrow^* v$

- That is, all programs terminate

So termination is trivially decidable (the constant “yes” function), so our language is not Turing-complete

The proof requires more advanced techniques than we have learned so far because the size of expressions and typing derivations does not decrease with each program step

- Could present it in about an hour if desired

Non-proof:

- Recursion in $\lambda$ calculus requires some sort of self-application
- Easy fact: For all $\Gamma$, $x$, and $\tau$, we cannot derive $\Gamma \vdash x \ x : \tau$