CSE-505: Programming Languages

Lecture 11 — STLC Extensions and Related Topics

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Review

\[ e ::= \lambda x. e \mid x \mid e e \mid c \]
\[ v ::= \lambda x. e \]
\[ \tau ::= \text{int} \mid \tau \rightarrow \tau \]
\[ \Gamma ::= \cdot \mid \Gamma, x : \tau \]

(\lambda x. e) v \rightarrow e[v/x]
\[ e_1 e_2 \rightarrow e'_1 e_2 \]
\[ v e_2 \rightarrow v e'_2 \]

\[ e[e'/x] : \text{capture-avoiding substitution of } e' \text{ for free } x \text{ in } e \]

\[ \Gamma \vdash c : \text{int} \]
\[ \Gamma \vdash x : \Gamma(x) \]
\[ \Gamma, x : \tau_1 \vdash e : \tau_2 \]

\[ \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 \]
\[ \Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \]
\[ \Gamma \vdash e_2 : \tau_2 \]

Preservation: If \( \cdot \vdash e : \tau \) and \( e \rightarrow e' \), then \( \cdot \vdash e' : \tau \).
Progress: If \( \cdot \vdash e : \tau \), then \( e \) is a value or \( \exists e' \) such that \( e \rightarrow e' \).

Adding Stuff

Time to use STLC as a foundation for understanding other common language constructs

We will add things via a principled methodology thanks to a proper education

- Extend the syntax
- Extend the operational semantics
  - Derived forms (syntactic sugar), or
  - Direct semantics
- Extend the type system
- Extend soundness proof (new stuck states, proof cases)

In fact, extensions that add new types have even more structure

Let bindings (CBV)

\[ e ::= \cdots \mid \text{let } x = e_1 \text{ in } e_2 \]

\[ \text{let } x = e_1 \text{ in } e_2 \rightarrow \text{let } x' = e'_1 \text{ in } e_2 \]

\[ \text{let } x = v \text{ in } e \rightarrow e[v/x] \]

\[ \Gamma \vdash e_1 : \tau' \]
\[ \Gamma, x : \tau' \vdash e_2 : \tau \]

\[ \Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau \]

(Also need to extend definition of substitution...)

Progress: If \( e \) is a let, 1 of the 2 new rules apply (using induction)

 Preservation: Uses Substitution Lemma

Substitution Lemma: Uses Weakening and Exchange
Derived forms

**let** seems just like $\lambda$, so can make it a derived form

- **let** $x = e_1$ in $e_2$ "a macro" / "desugars to" $(\lambda x. e_2) e_1$
- A “derived form”

(Harder if $\lambda$ needs explicit type)

Or just define the semantics to replace let with $\lambda$:

$$\text{let } x = e_1 \text{ in } e_2 \rightarrow (\lambda x. e_2) e_1$$

These 3 semantics are **different** in the state-sequence sense
$(e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_n)$

- But (totally) **equivalent** and you could prove it (not hard)

Booleans and Conditionals

$e ::= \cdots | true | false | if \ e_1 \ e_2 \ e_3$
$v ::= \cdots | true | false$
$\tau ::= \cdots | bool$

$$\frac{e_1 \rightarrow e_1'}{if \ e_1 \ e_2 \ e_3 \rightarrow if \ e_1' e_2 e_3}$$

$\Gamma \vdash e_1 : bool \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau$

$\Gamma \vdash true : bool \quad \Gamma \vdash false : bool$

Also extend definition of substitution (will stop writing that)...
Notes: CBN, new Canonical Forms case, all lemma cases easy

Pairs (CBV, left-right)

$$e ::= \cdots | (e, e) | e.1 | e.2$$
$$v ::= \cdots | (v, v)$$
$$\tau ::= \cdots | \tau \ast \tau$$

$e_1 \rightarrow e_1'$
$(e_1, e_2) \rightarrow (e_1', e_2)$

$e \rightarrow e'$
$(v_1, e_2) \rightarrow (v_1, e_2')$

Small-step can be a pain

- Large-step needs only 3 rules
- Will learn more concise notation later (evaluation contexts)

Pairs continued

$$\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2$$

$$\Gamma \vdash (e_1, e_2) : \tau_1 \ast \tau_2$$

$\Gamma \vdash e : \tau_1 \ast \tau_2$
$\Gamma \vdash e : \tau_1 \ast \tau_2$
$\Gamma \vdash e_1 : \tau_1$
$\Gamma \vdash e_2 : \tau_2$

Canonical Forms: If $\Gamma \vdash v : \tau_1 \ast \tau_2$, then $v$ has the form $(v_1, v_2)$

Progress: New cases using Canonical Forms are $v.1$ and $v.2$

Preservation: For primitive reductions, inversion gives the result **directly**
Records

Records are like $n$-ary tuples except with named fields

- Field names are not variables; they do not $\alpha$-convert

$\begin{align*}
e & ::= \cdots \mid \{l_1 = e_1 ; \ldots ; l_n = e_n\} \mid e.l \\
v & ::= \cdots \mid \{l_1 = v_1 ; \ldots ; l_n = v_n\} \\
\tau & ::= \cdots \mid \{l_1 : \tau_1 ; \ldots ; l_n : \tau_n\}
\end{align*}$

$\frac{e_i \rightarrow e'_i}{\{l_1 = v_1 , \ldots , l_{i-1} = v_{i-1} , l_i = e_i , \ldots , l_n = e_n\} \rightarrow \{l_1 = v_1 , \ldots , l_{i-1} = v_{i-1} , l_i = e'_i , \ldots , l_n = e_n\}}$

$\frac{\Gamma \vdash e_1 : \tau_1 \quad \cdots \quad \Gamma \vdash e_n : \tau_n \quad \text{labels distinct}}{\Gamma \vdash \{l_1 = e_1 , \ldots , l_n = e_n\} : \{l_1 : \tau_1 , \ldots , l_n : \tau_n\}}$

$\frac{\Gamma \vdash e : \{l_1 : \tau_1 , \ldots , l_n : \tau_n\} \quad 1 \leq i \leq n}{\Gamma \vdash e.l_i : \tau_i}$

Should we be allowed to reorder fields?

- $\cdot \vdash \{l_1 = 42 ; l_2 = \text{true}\} : \{l_2 : \text{bool} ; l_1 : \text{int}\}$ ??

Really a question about, “when are two types equal?”

Nothing wrong with this from a type-safety perspective, yet many languages disallow it

- Reasons: Implementation efficiency, type inference

Return to this topic when we study subtyping

Sums

What about ML-style datatypes:

$\text{type } t = A \mid B \of \text{int} \mid C \of \text{int} \ast t$

1. Tagged variants (i.e., discriminated unions)

2. Recursive types

3. Type constructors (e.g., type $'a\ mylist = \ldots$)

4. Named types

For now, just model (1) with (anonymous) sum types

- (2) is in a later lecture, (3) is straightforward, and (4) we’ll discuss informally

$\begin{align*}
e & ::= \cdots \mid A(e) \mid B(e) \mid \text{match } e \text{ with } A.x . e \mid B.x . e \\
v & ::= \cdots \mid A(v) \mid B(v) \\
\tau & ::= \cdots \mid \tau_1 + \tau_2
\end{align*}$

- Only two constructors: $A$ and $B$

- All values of any sum type built from these constructors

- So $A(e)$ can have any sum type allowed by $e$’s type

- No need to declare sum types in advance

- Like functions, will “guess the type” in our rules
Sums operational semantics

\[
\begin{align*}
\text{match } A(v) \text{ with } Ax. \, e_1 \mid By. \, e_2 & \rightarrow e_1[v/x] \\
\text{match } B(v) \text{ with } Ax. \, e_1 \mid By. \, e_2 & \rightarrow e_2[v/y]
\end{align*}
\]

\[
\begin{align*}
e & \rightarrow e' \\
A(e) & \rightarrow A(e') \\
B(e) & \rightarrow B(e')
\end{align*}
\]

\[
\begin{align*}
e & \rightarrow e' \\
\text{match } e \text{ with } Ax. \, e_1 \mid By. \, e_2 & \rightarrow \text{match } e' \text{ with } Ax. \, e_1 \mid By. \, e_2
\end{align*}
\]

\text{match has binding occurrences, just like pattern-matching}

(Definition of substitution must avoid capture, just like functions)

What is going on

Feel free to think about \textit{tagged values} in your head:

- A tagged value is a pair of:
  - A tag A or B (or 0 or 1 if you prefer)
  - The (underlying) value

- A match:
  - Checks the tag
  - Binds the variable to the (underlying) value

This much is just like OCaml and related to homework 2

Sums Typing Rules

Inference version (not trivial to infer; can require annotations)

\[
\begin{align*}
\Gamma & \vdash e : \tau_1 \\
\Gamma & \vdash A(e) : \tau_1 + \tau_2 \\
\Gamma & \vdash B(e) : \tau_1 + \tau_2
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash e : \tau_1 + \tau_2 \\
\Gamma, x: \tau_1 & \vdash e_1 : \tau \\
\Gamma, y: \tau_2 & \vdash e_2 : \tau
\end{align*}
\]

\[
\Gamma \vdash \text{match } e \text{ with } Ax. \, e_1 \mid By. \, e_2 : \tau
\]

Key ideas:

- For constructor-uses, “other side can be anything”
- For \texttt{match}, both sides need same type
  - Don’t know which branch will be taken, just like an \texttt{if}.
  - In fact, can drop explicit booleans and encode with sums:
    
    \texttt{E.g., bool = int + int, true = A(0), false = B(0)}

Sums Type Safety

Canonical Forms: If \( \cdot \vdash v : \tau_1 + \tau_2 \), then there exists a \( v_1 \) such that either \( v = A(v_1) \) and \( \cdot \vdash v_1 : \tau_1 \) or \( v = B(v_1) \) and \( \cdot \vdash v_1 : \tau_2 \)

- Progress for \texttt{match } v \texttt{ with } Ax. \, e_1 \mid By. \, e_2 \texttt{ follows, as usual, from Canonical Forms}
- Preservation for \texttt{match } v \texttt{ with } Ax. \, e_1 \mid By. \, e_2 \texttt{ follows from the type of the underlying value and the Substitution Lemma}
- The Substitution Lemma has new “hard” cases because we have new binding occurrences
- But that’s all there is to it (plus lots of induction)
What are sums for?

- Pairs, structs, records, aggregates are fundamental data-builders
- Sums are just as fundamental: “this or that not both”
- You have seen how OCaml does sums (datatypes)
- Worth showing how C and Java do the same thing
  - A primitive in one language is an idiom in another

Sums in C

type t = A of t1 | B of t2 | C of t3
match e with A x -> ...

One way in C:

```c
struct t {
    enum {A, B, C} tag;
    union {t1 a; t2 b; t3 c;} data;
};
... switch(e->tag){ case A: t1 x=e->data.a; ...}
```

- No static checking that tag is obeyed
- As fat as the fattest variant (avoidable with casts)
- Mutation costs us again!

Sums in Java

type t = A of t1 | B of t2 | C of t3
match e with A x -> ...

One way in Java (t4 is the match-expression’s type):

```java
abstract class t {abstract t4 m();}
class A extends t { t1 x; t4 m(){...}}
class B extends t { t2 x; t4 m(){...}}
class C extends t { t3 x; t4 m(){...}}
... e.m() ...
```

- A new method in t and subclasses for each match expression
- Supports extensibility via new variants (subclasses) instead of extensibility via new operations (match expressions)

Pairs vs. Sums

You need both in your language

- With only pairs, you clumsily use dummy values, waste space, and rely on unchecked tagging conventions
- Example: replace `int + (int -> int)` with `int * (int * (int -> int))`

Pairs and sums are “logical duals” (more on that later)

- To make a `τ1 * τ2` you need a `τ1` and a `τ2`
- To make a `τ1 + τ2` you need a `τ1` or a `τ2`
- Given a `τ1 * τ2`, you can get a `τ1` or a `τ2` (or both; your “choice”)
- Given a `τ1 + τ2`, you must be prepared for either a `τ1` or `τ2` (the value’s “choice”)
Recursion

We won’t prove it, but every extension so far preserves termination.

A Turing-complete language needs some sort of loop, but our
lambda-calculus encoding won’t type-check, nor will any encoding
of equal expressive power.

▶ So instead add an explicit construct for recursion.
▶ You might be thinking \textbf{let rec }\ f\ x = e, but we will do
something more concise and general but less intuitive.

We won’t prove it, but every extension so far preserves termination.

Base Types and Primitives, in general

What about floats, strings, …?
Could add them all or do something more general…

Parameterize our language/semantics by a collection of \textit{base types} \((b_1, \ldots, b_n)\) and \textit{primitives} \((p_1 : \tau_1, \ldots, p_n : \tau_n)\). Examples:

▶ \texttt{concat : string→string→string}
▶ \texttt{toInt : float→int}
▶ “hello” : string

For each primitive, assume if applied to values of the right types it
produces a value of the right type.

Together the types and assumed steps tell us how to type-check
and evaluate \(p_i\ v_1 \ldots v_n\) where \(p_i\) is a primitive.

We can prove soundness \textit{once and for all} given the assumptions.

Recursion

To use \textbf{fix} like \textbf{let rec}, just pass it a two-argument function where
the first argument is for recursion.

▶ Not shown: \textbf{fix} and tuples can also encode mutual recursion.

Example:
\[
\textbf{fix }\lambda f.\lambda n.\text{ if } (n<1)\ 1\ (n \ast (f(n - 1)))\ 5
\]

No new values and no new types.
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion
- Not shown: fix and tuples can also encode mutual recursion

Example:
$$
\begin{align*}
\&( \text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))) 5 \\
\rightarrow & (\lambda n. \text{ if } (n < 1) 1 (n * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(n - 1)))) 5 \\
\rightarrow & \text{if } (5 < 1) 1 (5 * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(5 - 1)) \\
\rightarrow & 2 \\
\rightarrow & 5 * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(5 - 1)
\end{align*}
$$

Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion
- Not shown: fix and tuples can also encode mutual recursion

Example:
$$
\begin{align*}
\&( \text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))) 5 \\
\rightarrow & (\lambda n. \text{ if } (n < 1) 1 (n * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(n - 1)))) 5 \\
\rightarrow & \text{if } (5 < 1) 1 (5 * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(5 - 1)) \\
\rightarrow & 2 \\
\rightarrow & 5 * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(5 - 1) \\
\rightarrow & 2 \\
\rightarrow & 5 * ((\lambda n. \text{ if } (n < 1) 1 (n * ((\text{fix } \lambda f. \lambda n. \text{ if } (n < 1) 1 (n * (f(n - 1)))))(n - 1)))) 4) \\
\rightarrow & ...
\end{align*}
$$
Why called fix?

In math, a fix-point of a function $g$ is an $x$ such that $g(x) = x$.

- This makes sense only if $g$ has type $\tau \rightarrow \tau$ for some $\tau$.
- A particular $g$ could have have 0, 1, 39, or infinity fix-points.
- Examples for functions of type $\text{int} \rightarrow \text{int}$:
  - $\lambda x. x + 1$ has no fix-points
  - $\lambda x. x \times 0$ has one fix-point
  - $\lambda x. \text{absolute\_value}(x)$ has an infinite number of fix-points
  - $\lambda x. \text{if}\ (x < 10 \&\& x > 0)\ x \ 0$ has 10 fix-points

Higher types

At higher types like $(\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})$, the notion of fix-point is exactly the same (but harder to think about).

- For what inputs $f$ of type $\text{int} \rightarrow \text{int}$ is $g(f) = f$?

Examples:

- $\lambda f. \lambda x. (f x) + 1$ has no fix-points
- $\lambda f. \lambda x. (f x) \times 0$ (or just $\lambda f. \lambda x. 0$) has 1 fix-point
  - The function that always returns 0
  - In math, there is exactly one such function (cf. equivalence)
- $\lambda f. \lambda x. \text{absolute\_value}(f x)$ has an infinite number of fix-points: Any function that never returns a negative result

Back to factorial

Now, what are the fix-points of $\lambda f. \lambda x. \text{if}\ (x < 1)\ 1\ (x \times (f(x - 1)))$?

It turns out there is exactly one (in math): the factorial function!

And $\text{fix} \ \lambda f. \lambda x. \text{if}\ (x < 1)\ 1\ (x \times (f(x - 1)))$ behaves just like the factorial function.

- That is, it behaves just like the fix-point of $\lambda f. \lambda x. \text{if}\ (x < 1)\ 1\ (x \times (f(x - 1)))$
- In general, $\text{fix}$ takes a function-taking-function and returns its fix-point

(This isn’t necessarily important, but it explains the terminology and shows that programming is deeply connected to mathematics)

Typing fix

$$\Gamma \vdash e : \tau \rightarrow \tau$$
$$\Gamma \vdash \text{fix} e : \tau$$

Math explanation: If $e$ is a function from $\tau$ to $\tau$, then $\text{fix} e$, the fixed-point of $e$, is some $\tau$ with the fixed-point property.

- So it’s something with type $\tau$.

Operational explanation: $\text{fix} \ \lambda x. e'$ becomes $e'[\text{fix} \ \lambda x. e'/x]$

- The substitution means $x$ and $\text{fix} \ \lambda x. e'$ need the same type
- The result means $e'$ and $\text{fix} \ \lambda x. e'$ need the same type

Note: The $\tau$ in the typing rule is usually insantiated with a function type.

- e.g., $\tau_1 \rightarrow \tau_2$, so $e$ has type $(\tau_1 \rightarrow \tau_2) \rightarrow (\tau_1 \rightarrow \tau_2)$

Note: Proving soundness is straightforward!
General approach

We added let, booleans, pairs, records, sums, and fix

- let was syntactic sugar
- fix made us Turing-complete by “baking in” self-application
- The others added types

Whenever we add a new form of type $\tau$ there are:

- Introduction forms (ways to make values of type $\tau$)
- Elimination forms (ways to use values of type $\tau$)

What are these forms for functions? Pairs? Sums?

When you add a new type, think “what are the intro and elim forms”?

Anonymity

We added many forms of types, all unnamed a.k.a. structural. Many real PLs have (all or mostly) named types:

- Java, C, C++: all record types (or similar) have names
  - Omitting them just means compiler makes up a name
- OCaml sum types and record types have names

A never-ending debate:

- Structural types allow more code reuse: good
- Named types allow less code reuse: good
- Structural types allow generic type-based code: good
- Named types let type-based code distinguish names: good

The theory is often easier and simpler with structural types

Termination

Surprising fact: If $\cdot \vdash e : \tau$ in STLC with all our additions except fix, then there exists a $v$ such that $e \rightarrow^* v$

- That is, all programs terminate

So termination is trivially decidable (the constant “yes” function), so our language is not Turing-complete

The proof requires more advanced techniques than we have learned so far because the size of expressions and typing derivations does not decrease with each program step

- Could present it in about an hour if desired

Non-proof:

- Recursion in $\lambda$ calculus requires some sort of self-application
- Easy fact: For all $\Gamma$, $x$, and $\tau$, we cannot derive $\Gamma \vdash x : \tau$