CSE-505: Programming Languages

Lecture 9 — Simply Typed Lambda Calculus

Zach Tatlock
2016
Types

Major new topic worthy of several lectures: Type systems
- Continue to use (CBV) Lambda Calculus as our core model
- But will soon enrich with other common primitives

This lecture:
- Motivation for type systems
- What a type system is designed to do and not do
  - Definition of stuckness, soundness, completeness, etc.
- The Simply-Typed Lambda Calculus
  - A basic and natural type system
  - Starting point for more expressiveness later

Next lecture:
- Prove Simply-Typed Lambda Calculus is sound
Review: L-R CBV Lambda Calculus

\[ e ::= \lambda x. e \mid x \mid e e \]

\[ v ::= \lambda x. e \]

Implicit systematic renaming of bound variables

- \( \alpha \)-equivalence on expressions ("the same term")

\[
\begin{align*}
&\quad e \rightarrow e' \\
&\quad (\lambda x. e) v \rightarrow e[v/x] \\
&\quad e_1 \ e_2 \rightarrow e'_1 \ e_2 \\
&\quad v \ e_2 \rightarrow v \ e'_2 \\
&\quad e_1[e_2/x] = e_3 \\
&\quad x[e/x] = e \\
&\quad y \neq x \\
&\quad y[e/x] = y \\
&\quad e_1[e/x] = e'_1 \\
&\quad e_2[e/x] = e'_2 \\
&\quad (e_1 \ e_2)[e/x] = e'_1 \ e'_2 \\
&\quad e_1[e/x] = e'_1 \\
&\quad y \neq x \\
&\quad y \not\in FV(e) \\
&\quad (\lambda y. e_1)[e/x] = \lambda y. e'_1
\end{align*}
\]
Introduction to Types

Naive thought: More powerful PLs are always better

- Be Turing Complete (e.g., Lambda Calculus or x86 Assembly)
- Have really flexible features (e.g., lambdas)
- Have conveniences to keep programs short

If this is the only metric, types are a step backward

- Whole point is to allow fewer programs
- A “filter” between abstract syntax and compiler/interpreter
  - Fewer programs in language means less for a correct implementation
- So if types are a great idea, they must help with other desirable properties for a PL...
Why types? (Part 1)

1. Catch “simple” mistakes early, even for untested code
   - Example: “if” applied to “mkpair”
   - Even if some too-clever programmer meant to do it
   - Even though decidable type systems must be conservative
Why types? (Part 1)

1. Catch “simple” mistakes early, even for untested code
   ▶ Example: “if” applied to “mkpair”
   ▶ Even if some too-clever programmer meant to do it
   ▶ Even though decidable type systems must be conservative

2. (Safety) Prevent getting stuck (e.g., $x \nu$)
   ▶ Ensure execution never gets to a “meaningless” state
   ▶ But “meaningless” depends on the semantics
   ▶ Each PL typically makes some things type errors (again being conservative) and others run-time errors
Why types? (Part 1)

1. Catch “simple” mistakes early, even for untested code
   ▶ Example: “if” applied to “mkpair”
   ▶ Even if some too-clever programmer meant to do it
   ▶ Even though decidable type systems must be conservative

2. (Safety) Prevent getting stuck (e.g., $x \ v$)
   ▶ Ensure execution never gets to a “meaningless” state
   ▶ But “meaningless” depends on the semantics
   ▶ Each PL typically makes some things type errors (again being conservative) and others run-time errors

3. Enforce encapsulation (an abstract type)
   ▶ Clients can’t break invariants
   ▶ Clients can’t assume an implementation
   ▶ Requires safety, meaning no “stuck” states that corrupt run-time (e.g., C/C++)
   ▶ Can enforce encapsulation without static types, but types are a particularly nice way
4. Assuming well-typedness allows faster implementations
   - Smaller interfaces enable optimizations
   - Don’t have to check for impossible states
   - Orthogonal to safety (e.g., C/C++)
Why types? (Part 2)

4. Assuming well-typedness allows faster implementations
   ▶ Smaller interfaces enable optimizations
   ▶ Don’t have to check for impossible states
   ▶ Orthogonal to safety (e.g., C/C++)

5. Syntactic overloading
   ▶ Have symbol lookup depend on operands’ types
   ▶ Only modestly interesting semantically
   ▶ Late binding (lookup via run-time types) more interesting
Why types? (Part 2)

4. Assuming well-typedness allows faster implementations
   ▶ Smaller interfaces enable optimizations
   ▶ Don’t have to check for impossible states
   ▶ Orthogonal to safety (e.g., C/C++)

5. Syntactic overloading
   ▶ Have symbol lookup depend on operands’ types
   ▶ Only modestly interesting semantically
   ▶ Late binding (lookup via *run-time* types) more interesting

6. Detect other errors via extensions
   ▶ Often via a “type-and-effect” system
   ▶ Deep similarities in analyses suggest type systems a good way to think-about/define/prove what you’re checking
   ▶ Uncaught exceptions, tainted data, non-termination, IO performed, data races, dangling pointers, ...
4. Assuming well-typedness allows faster implementations
   ▶ Smaller interfaces enable optimizations
   ▶ Don’t have to check for impossible states
   ▶ Orthogonal to safety (e.g., C/C++)

5. Syntactic overloading
   ▶ Have symbol lookup depend on operands’ types
   ▶ Only modestly interesting semantically
   ▶ Late binding (lookup via run-time types) more interesting

6. Detect other errors via extensions
   ▶ Often via a “type-and-effect” system
   ▶ Deep similarities in analyses suggest type systems a good way to think-about/define/prove what you’re checking
   ▶ Uncaught exceptions, tainted data, non-termination, IO performed, data races, dangling pointers, ...

We’ll focus on (1), (2), and (3) and maybe (6)
What is a type system?

Er, uh, you know it when you see it. Some clues:

- A decidable (?) judgment for classifying programs
  - E.g., $e_1 + e_2$ has type int if $e_1$, $e_2$ have type int (else no type)
- A sound (?) abstraction of computation
  - E.g., if $e_1 + e_2$ has type int, then evaluation produces an int (with caveats!)
- Fairly syntax directed
  - Non-example (?): $e$ terminates within 100 steps
- Particularly fuzzy distinctions with abstract interpretation
  - Possible topic for a later lecture
  - Often a more natural framework for flow-sensitive properties
  - Types often more natural for higher-order programs

This is a CS-centric, PL-centric view. Foundational type theory has more rigorous answers

- Later lecture: Typed PLs are like proof systems for logics
Plan for 3ish weeks

- Simply typed $\lambda$ calculus
- (Syntactic) Type Soundness (i.e., safety)
- Extensions (pairs, sums, lists, recursion)

Break for the Curry-Howard isomorphism; continuations; midterm

- Subtyping
- Polymorphic types (generics)
- Recursive types
- Abstract types
- Effect systems

Homework: Adding back mutation
Omitted: Type inference
Adding constants

Enrich the Lambda Calculus with integer constants:

- Not strictly necessary, but makes types seem more natural

\[
e ::= \lambda x. e \mid x \mid e\,e \mid c
\]

\[
v ::= \lambda x. e \mid c
\]

*No new operational-semantics rules since constants are values*

We could add + and other *primitives*

- Then we would need new rules (e.g., 3 small-step for +)
- Alternately, parameterize “programs” by primitives:
  \[
  \lambda \text{plus}. \lambda \text{times}. \ldots. e
  \]
  - Like Pervasives in OCaml
  - A great way to keep language definitions small
Stuck

Key issue: can a program “get stuck” (reach a “bad” state)?

- Definition: *e is stuck* if *e* is not a value and there is no *e’* such that *e* $\rightarrow$ *e’*

- Definition: *e can get stuck* if there exists an *e’* such that *e* $\rightarrow^*$ *e’* and *e’* is stuck
  - In a deterministic language, *e* “gets stuck”

Most people don’t appreciate that stuckness depends on the operational semantics
  - Inherent given the definitions above
What’s stuck?

Given our language, what are the set of stuck expressions?

- Note: Explicitly defining the stuck states is unusual

\[
e ::= \lambda x. e \mid x \mid e e \mid c
\]

\[
v ::= \lambda x. e \mid c
\]

\[
(\lambda x. e) v \rightarrow e[v/x]
\]

\[
\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2}
\]

\[
\frac{e_2 \rightarrow e'_2}{v e_2 \rightarrow v e'_2}
\]

(Hint: The full set is recursively defined.)

\[
S ::= 
\]
What’s stuck?

Given our language, what are the set of stuck expressions?

Note: Explicitly defining the stuck states is unusual

\[
e ::= \lambda x. e \mid x \mid e \ e \mid c
\]

\[
v ::= \lambda x. e \mid c
\]

\[
\begin{align*}
(\lambda x. \ e) \ v & \rightarrow \ e[v/x] \\
{\frac{e_1 \rightarrow e'_1}{e_1 \ e_2 \rightarrow e'_1 \ e_2}} & \quad {\frac{e_2 \rightarrow e'_2}{v \ e_2 \rightarrow v \ e'_2}}
\end{align*}
\]

(Hint: The full set is recursively defined.)

\[
S ::= x \mid c \ v \mid S \ e \mid v \ S
\]
What’s stuck?

Given our language, what are the set of stuck expressions?

- Note: Explicitly defining the stuck states is unusual

\[
e ::= \lambda x. e \mid x \mid e e \mid c
\]

\[
v ::= \lambda x. e \mid c
\]

\[
(\lambda x. e) v \rightarrow e[v/x]
\]

\[
e_1 \rightarrow e'_1
\]

\[
e_1 e_2 \rightarrow e'_1 e_2
\]

\[
e_2 \rightarrow e'_2
\]

\[
v e_2 \rightarrow v e'_2
\]

(Hint: The full set is recursively defined.)

\[
S ::= x \mid c v \mid S e \mid v S
\]

Note: Can have fewer stuck states if we add more rules

- Example: Javascript

- Example: \[c v \rightarrow v\]

- In unsafe languages, stuck states can set the computer on fire
Soundness and Completeness

A *type system* is a judgment for classifying programs

- “accepts” a program if some complete derivation gives it a type, else “rejects”

A *sound* type system never accepts a program that can get stuck

- No false negatives

A *complete* type system never rejects a program that can’t get stuck

- No false positives

It is typically *undecidable* whether a stuck state can be reachable

- Corollary: If we want an *algorithm* for deciding if a type system accepts a program, then the type system cannot be sound and complete

- We’ll choose soundness, try to reduce false positives in practice
Wrong Attempt

\[ \tau ::= \text{int} \mid \text{fn} \]

\[ \vdash e : \tau \]

\[ \vdash \lambda x. e : \text{fn} \quad \vdash c : \text{int} \quad \vdash e_1 : \text{fn} \quad \vdash e_2 : \text{int} \]

\[ \vdash e_1 e_2 : \text{int} \]
Wrong Attempt

\[ \tau ::= \text{int} \mid \text{fn} \]

\[ \vdash e : \tau \]

\[ \begin{array}{c}
\vdash \lambda x. e : \text{fn} \\
\vdash c : \text{int}
\end{array} \quad \begin{array}{c}
\vdash e_1 : \text{fn} \\
\vdash e_2 : \text{int}
\end{array} \quad \vdash e_1 \, e_2 : \text{int} \]

1. NO: can get stuck, e.g., \((\lambda x. y) \, 3\)

2. NO: too restrictive, e.g., \((\lambda x. x \, 3) \, (\lambda y. y)\)

3. NO: types not preserved, e.g., \((\lambda x. \lambda y. y) \, 3\)
Getting it right

1. Need to type-check function bodies, which have free variables
2. Need to classify functions using argument and result types

For (1): $\Gamma ::= \cdot \mid \Gamma, x : \tau$ and $\Gamma \vdash e : \tau$
   ▶ Require whole program to type-check under empty context $\cdot$

For (2): $\tau ::= \textbf{int} \mid \tau \rightarrow \tau$
   ▶ An infinite number of types:
     $\textbf{int} \rightarrow \textbf{int}$, $(\textbf{int} \rightarrow \textbf{int}) \rightarrow \textbf{int}$, $\textbf{int} \rightarrow (\textbf{int} \rightarrow \textbf{int})$, ...

Concrete syntax note: $\rightarrow$ is right-associative, so
$\tau_1 \rightarrow \tau_2 \rightarrow \tau_3$ is $\tau_1 \rightarrow (\tau_2 \rightarrow \tau_3)$
STLC Type System

\[ \tau ::= \text{int} \mid \tau \rightarrow \tau \]
\[ \Gamma ::= \cdot \mid \Gamma, x:\tau \]

\[ \Gamma \vdash e : \tau \]

\[ \Gamma \vdash c : \text{int} \quad \Gamma \vdash x : \Gamma(x) \]

\[ \Gamma, x : \tau_1 \vdash e : \tau_2 \]
\[ \Gamma \vdash \lambda x. \ e : \tau_1 \rightarrow \tau_2 \]

The function-introduction rule is the interesting one...
A closer look

\[ \Gamma, x : \tau_1 \vdash e : \tau_2 \]
\[ \frac{}{\Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2} \]

Where did \( \tau_1 \) come from?

- Our rule “inferred” or “guessed” it
- To be syntax directed, change \( \lambda x. e \) to \( \lambda x : \tau. e \) and use that \( \tau \)

Can think of “adding \( x \)” as shadowing or requiring \( x \not\in \text{Dom}(\Gamma) \)

- Systematic renaming (\( \alpha \)-conversion) ensures \( x \not\in \text{Dom}(\Gamma) \) is not a problem
A closer look

\[
\Gamma, x : \tau_1 \vdash e : \tau_2 \\
\frac{}{\Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2}
\]

Is our type system too restrictive?

▶ That’s a matter of opinion
▶ But it does reject programs that don’t get stuck

Example: \((\lambda x. (x (\lambda y. y)) (x 3)) \lambda z. z\)

▶ Does not get stuck: Evaluates to 3
▶ Does not type-check:
  ▶ There is no \(\tau_1, \tau_2\) such that \(x : \tau_1 \vdash (x (\lambda y. y)) (x 3) : \tau_2\)
    because you have to pick one type for \(x\)
Always restrictive

Whether or not a program “gets stuck” is undecidable:
- If $e$ has no constants or free variables, then $e \ (3 \ 4)$ or $e \ x$ gets stuck if and only if $e$ terminates (cf. the halting problem)

Old conclusion: “Strong types for weak minds”
- Need a back door (unchecked casts)

Modern conclusion: Unsafe constructs almost never worth the risk
- Make “false positives” (rejecting safe program) rare enough
  - Have compile-time resources for “fancy” type systems
- Make workarounds for false positives convenient enough
How does STLC measure up?

So far, STLC is sound:

▶ As language dictators, we decided $c v$ and undefined variables were “bad” meaning neither values nor reducible
▶ Our type system is a conservative checker that an expression will never get stuck

But STLC is far too restrictive:

▶ In practice, just too often that it prevents safe and natural code reuse
▶ More fundamentally, it’s not even Turing-complete
  ▶ Turns out all (well-typed) programs terminate
  ▶ A good-to-know and useful property, but inappropriate for a general-purpose PL
  ▶ That’s okay: We will add more constructs and typing rules
Type Soundness

We will take a *syntactic* (operational) approach to soundness/safety

- The popular way since the early 1990s

Theorem (Type Safety): If \( \cdot \vdash e : \tau \) then \( e \) diverges or \( e \rightarrow^n v \) for an \( n \) and \( v \) such that \( \cdot \vdash v : \tau \)

- That is, if \( \cdot \vdash e : \tau \), then \( e \) cannot get stuck

Proof: Next lecture