Where we are

- Done: OCaml tutorial, “IMP” syntax, structural induction
- Now: Operational semantics for our little “IMP” language
  - Most of what you need for Homework 1
  - (But Problem 4 requires proofs over semantics)
IMP’s abstract syntax is defined inductively:

\[
\begin{align*}
  s &::= \text{skip} | x := e | s; s | \text{if } e \ s \ s | \text{while } e \ s \\
  e &::= c | x | e + e | e * e \\
  (c &\in \{\ldots, -2, -1, 0, 1, 2, \ldots \}) \\
  (x &\in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \})
\end{align*}
\]

We haven’t yet said what programs mean! (Syntax is boring)

Encode our “social understanding” about variables and control flow
Outline

- Semantics for expressions
  1. Informal idea; the need for *heaps*
  2. Definition of heaps
  3. The evaluation *judgment* (a relation form)
  4. The evaluation *inference rules* (the relation definition)
  5. Using inference rules
     - *Derivation trees* as interpreters
     - Or as *proofs* about expressions
  6. *Metatheory*: Proofs about the semantics
- Then semantics for statements
  - ...
Informal idea

Given \( e \), what \( c \) does \( e \) evaluate to?

\[
1 + 2 \quad x + 2
\]
Informal idea

Given $e$, what $c$ does $e$ evaluate to?

$$1 + 2 \quad x + 2$$

It depends on the values of variables (of course)

Use a heap $H$ for a total function from variables to constants

- Could use partial functions, but then $\exists H$ and $e$ for which there is no $c$

We’ll define a relation over triples of $H$, $e$, and $c$

- Will turn out to be function if we view $H$ and $e$ as inputs and $c$ as output
- With our metalanguage, easier to define a relation and then prove it is a function (if, in fact, it is)
Heaps

\[ H ::= \cdot \mid H, x \mapsto c \]

A lookup-function for heaps:

\[ H(x) = \begin{cases} 
  c & \text{if } H = H', x \mapsto c \\
  H'(x) & \text{if } H = H', y \mapsto c' \text{ and } y \neq x \\
  0 & \text{if } H = \cdot \end{cases} \]

- Last case avoids “errors” (makes function total)

“What heap to use” will arise in the semantics of statements

- For expression evaluation, “we are given an H”
The judgment

We will write: \[
H ; e \downarrow c
\]
to mean, “\(e\) evaluates to \(c\) under heap \(H\)”

It is just a relation on triples of the form \((H, e, c)\)

We just made up metasyntax \(H ; e \downarrow c\) to follow PL convention and to distinguish it from other relations

We can write: \(., x \mapsto 3 ; x + y \downarrow 3\), which will turn out to be \textit{true}\n(twoo triple will be in the relation we define)

Or: \(., x \mapsto 3 ; x + y \downarrow 6\), which will turn out to be \textit{false}\n(twoo triple will not be in the relation we define)
### Inference rules

#### Const

\[
\begin{align*}
\text{CONST} & \\
H & ; c \Downarrow c
\end{align*}
\]

#### Var

\[
\begin{align*}
\text{VAR} & \\
H & ; x \Downarrow H(x)
\end{align*}
\]

#### Add

\[
\begin{align*}
\text{ADD} & \\
H & ; e_1 \Downarrow c_1 \quad H & ; e_2 \Downarrow c_2 \\
\hline
H & ; e_1 + e_2 \Downarrow c_1 + c_2
\end{align*}
\]

#### Mult

\[
\begin{align*}
\text{MULT} & \\
H & ; e_1 \Downarrow c_1 \quad H & ; e_2 \Downarrow c_2 \\
\hline
H & ; e_1 \ast e_2 \Downarrow c_1 \ast c_2
\end{align*}
\]

**Top:** hypotheses  
**Bottom:** conclusion (read first)

By definition, if all hypotheses hold, then the conclusion holds

Each rule is a *schema* you “instantiate consistently”

- So rules “work” “for all” $H$, $c$, $e_1$, etc.
- But “each” $e_1$ has to be the “same” expression
Instantiating rules

Example instantiation:

\[ \cdot, y \mapsto 4 \; ; \; 3 + y \Downarrow 7 \]
\[ \cdot, y \mapsto 4 \; ; \; 5 \Downarrow 5 \]
\[ \cdot, y \mapsto 4 \; ; \; (3 + y) + 5 \Downarrow 12 \]

Instantiates:

\[ \text{ADD} \]
\[ H \; ; \; e_1 \Downarrow c_1 \quad H \; ; \; e_2 \Downarrow c_2 \]
\[ H \; ; \; e_1 + e_2 \Downarrow c_1 + c_2 \]

with

\[ H = \cdot, y \mapsto 4 \]
\[ e_1 = (3 + y) \]
\[ c_1 = 7 \]
\[ e_2 = 5 \]
\[ c_2 = 5 \]
A (complete) derivation is a tree of instantiations with axioms at the leaves.

Example:

\[
\begin{align*}
\cdot, y \mapsto 4 ; 3 & \Downarrow 3 \\
\cdot, y \mapsto 4 ; y & \Downarrow 4 \\
\cdot, y \mapsto 4 ; 3 + y & \Downarrow 7 \\
\cdot, y \mapsto 4 ; (3 + y) + 5 & \Downarrow 12
\end{align*}
\]

By definition, \( H ; e \Downarrow c \) if there exists a derivation with \( H ; e \Downarrow c \) at the root.
Back to relations

So what relation do our inference rules define?

- Start with empty relation (no triples) $R_0$

- Let $R_i$ be $R_{i-1}$ union all $H; e \downarrow c$ such that we can instantiate some inference rule to have conclusion $H; e \downarrow c$ and all hypotheses in $R_{i-1}$
  - So $R_i$ is all triples at the bottom of height-$j$ complete derivations for $j \leq i$

- $R_\infty$ is the relation we defined
  - All triples at the bottom of complete derivations

For the math folks: $R_\infty$ is the smallest relation closed under the inference rules
What are these things?

We can view the inference rules as defining an *interpreter*

- Complete derivation shows recursive calls to the “evaluate expression” function
  - Recursive calls from conclusion to hypotheses
  - *Syntax-directed* means the interpreter need not “search”

- See OCaml code in Homework 1

Or we can view the inference rules as defining a *proof system*

- Complete derivation proves facts from other facts starting with axioms
  - Facts established from hypotheses to conclusions
Some theorems

- Progress: For all $H$ and $e$, there exists a $c$ such that $H ; e \downarrow c$

- Determinacy: For all $H$ and $e$, there is at most one $c$ such that $H ; e \downarrow c$

We rigged it that way...
what would division, undefined-variables, or gettime() do?

Proofs are by induction on the the structure (i.e., height) of the expression $e$
On to statements

A statement does not produce a constant
On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.
  ▶ If it terminates
On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.

> If it terminates

We could define $H_1 ; s \Downarrow H_2$

> Would be a partial function from $H_1$ and $s$ to $H_2$

> Works fine; could be a homework problem
On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.
  ▶ If it terminates

We could define $H_1 ; s \downarrow H_2$
  ▶ Would be a partial function from $H_1$ and $s$ to $H_2$
  ▶ Works fine; could be a homework problem

Instead we’ll define a “small-step” semantics and then “iterate” to “run the program”

\[ H_1 ; s_1 \rightarrow H_2 ; s_2 \]
Statement semantics

\[ H_1 ; s_1 \rightarrow H_2 ; s_2 \]

ASSIGN

\[
\frac{H ; e \downarrow c}{H ; x := e \rightarrow H, x \leftarrow c ; \text{skip}}
\]

SEQ\(1\)

\[
\frac{H ; \text{skip}; s \rightarrow H ; s}{H ; \text{skip}; s \rightarrow H ; s}
\]

IF\(1\)

\[
\frac{H ; e \downarrow c \quad c > 0}{H ; \text{if } e \ s_1 \ s_2 \rightarrow H ; s_1}
\]

SEQ\(2\)

\[
\frac{H ; s_1 \rightarrow H' ; s'_1}{H ; s_1; s_2 \rightarrow H' ; s'_1; s_2}
\]

IF\(2\)

\[
\frac{H ; e \downarrow c \quad c \leq 0}{H ; \text{if } e \ s_1 \ s_2 \rightarrow H ; s_2}
\]
What about \textbf{while} \( e \) \( s \) (do \( s \) and loop if \( e > 0 \))?
Statement semantics cont’d

What about \textbf{while }e \; s \; (\text{do } s \text{ and loop if } e > 0)\text{?}

\[
\text{WHILE} \\
H \; ; \text{while } e \; s \rightarrow H \; ; \text{if } e \; (s; \text{while } e \; s) \; \text{skip}
\]

Many other equivalent definitions possible
Program semantics

Defined \( H ; s \rightarrow H' ; s' \), but what does “s” mean/do?

Our machine iterates: \( H_1 ; s_1 \rightarrow H_2 ; s_2 \rightarrow H_3 ; s_3 \ldots \),
with each step justified by a complete derivation using our single-step statement semantics

Let \( H_1 ; s_1 \rightarrow^n H_2 ; s_2 \) mean “becomes after n steps”

Let \( H_1 ; s_1 \rightarrow^* H_2 ; s_2 \) mean “becomes after 0 or more steps”

Pick a special “answer” variable \( \text{ans} \)

The program \( s \) produces \( c \) if \( ; s \rightarrow^* H ; \text{skip} \) and \( H(\text{ans}) = c \)

Does every \( s \) produce a \( c \)?
Example program execution

\[ \begin{align*}
x & := 3; (y := 1; \textbf{while} \ x \ (y := y \ast x; x := x - 1)) \\
\end{align*} \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \ast x; x := x - 1) \).
Example program execution

\[ x := 3; (y := 1; \textbf{while} \ x (y := y \times x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[
\cdot; \ x := 3; \ y := 1; \textbf{while} \ x \ s
\]
Example program execution

\[ x := 3; (y := 1; \textbf{while } x (y := y \times x; x := x - 1)) \]

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\[ \cdot ; x := 3; y := 1; \textbf{while } x \ s \]

\[ \rightarrow \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while } x \ s \]
Example program execution

\[ x := 3; (y := 1; \textbf{while } x (y := y \ast x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \ast x; x := x - 1) \).

\[ \cdot; x := 3; y := 1; \textbf{while } x \ s \]

\[ \rightarrow \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while } x \ s \]

\[ \rightarrow \cdot, x \mapsto 3; y := 1; \textbf{while } x \ s \]
Example program execution

\[ x := 3; (y := 1; \textbf{while} x (y := y \ast x; x := x-1)) \]

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\[
\begin{align*}
\cdot; x := 3; y := 1; \textbf{while} x s \\
\rightarrow & \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while} x s \\
\rightarrow & \cdot, x \mapsto 3; y := 1; \textbf{while} x s \\
\rightarrow^2 & \cdot, x \mapsto 3, y \mapsto 1; \textbf{while} x s
\end{align*}
\]
Example program execution

\[ x := 3; (y := 1; \text{while } x \ (y := y \times x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[ \cdot; x := 3; y := 1; \text{while } x \ s \]
\[ \rightarrow \quad \cdot, x \mapsto 3; \text{skip}; y := 1; \text{while } x \ s \]
\[ \rightarrow \quad \cdot, x \mapsto 3; y := 1; \text{while } x \ s \]
\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1; \text{while } x \ s \]
\[ \rightarrow \quad \cdot, x \mapsto 3, y \mapsto 1; \text{if } x \ (s; \text{while } x \ s) \ \text{skip} \]
Example program execution

\[ x := 3; (y := 1; \textbf{while } x \ (y := y \times x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[
\begin{align*}
\cdot; \ x & := 3; \ y := 1; \textbf{while } x \ s \\
\rightarrow & \quad \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while } x \ s \\
\rightarrow & \quad \cdot, x \mapsto 3; \ y := 1; \textbf{while } x \ s \\
\rightarrow^2 & \quad \cdot, x \mapsto 3, y \mapsto 1; \textbf{while } x \ s \\
\rightarrow & \quad \cdot, x \mapsto 3, y \mapsto 1; \textbf{if } x \ (s; \textbf{while } x \ s) \textbf{ skip} \\
\rightarrow & \quad \cdot, x \mapsto 3, y \mapsto 1; \ y := y \times x; x := x - 1; \textbf{while } x \ s
\end{align*}
\]
Continued...

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; \quad x := x - 1; \quad \textbf{while} \quad x \quad \textbf{s} \]
Continued...

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; \quad x := x - 1; \textbf{while} \ x \ s \]

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \textbf{while} \ x \ s \]
Continued...

\[ \rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; \ x := x-1; \textbf{while } x \ s \]

\[ \rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \textbf{while } x \ s \]

\[ \rightarrow \ldots, y \mapsto 3, x \mapsto 2; \textbf{if } x \ (s; \textbf{while } x \ s) \ \textbf{skip} \]
Continued...

\[\rightarrow^2 (\cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; x := x - 1; \text{while} \ x \ s)\]

\[\rightarrow^2 (\cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \text{while} \ x \ s)\]

\[\rightarrow (\cdot, y \mapsto 3, x \mapsto 2; \text{if} \ x \ (s; \text{while} \ x \ s) \ \text{skip})\]

\[\ldots\]

\[\ldots\]
Continued...

\[\begin{align*}
\rightarrow^2 & ., x \mapsto 3, y \mapsto 1, y \mapsto 3; \ x := x-1; \ \textbf{while} \ x \ s \\
\rightarrow^2 & ., x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \ \textbf{while} \ x \ s \\
\rightarrow & \ldots, y \mapsto 3, x \mapsto 2; \ \textbf{if} \ x (s; \ \textbf{while} \ x \ s) \ \textbf{skip} \\
\ldots & \\
\rightarrow & \ldots, y \mapsto 6, x \mapsto 0; \ \textbf{skip}
\end{align*}\]
Where we are

Defined $H ; e \downarrow c$ and $H ; s \rightarrow H' ; s'$ and extended the latter to give $s$ a meaning

- The way we did expressions is “large-step operational semantics”
- The way we did statements is “small-step operational semantics”
- So now you have seen both

Definition by interpretation: program means what an interpreter (written in a metalanguage) says it means

- Interpreter represents a (very) abstract machine that runs code

Large-step does not distinguish errors and divergence

- But we defined IMP to have no errors
- And expressions never diverge
Establishing Properties

We can prove a property of a terminating program by “running” it.

Example: Our last program terminates with $x$ holding 0.
Establishing Properties

We can prove a property of a terminating program by “running” it

Example: Our last program terminates with \( x \) holding 0

We can prove a program diverges, i.e., for all \( H \) and \( n \),
\[
\cdot; s \rightarrow^n H ; \text{skip}
\]
cannot be derived

Example: \textbf{while 1 skip}
Establishing Properties

We can prove a property of a terminating program by “running” it

Example: Our last program terminates with $x$ holding 0

We can prove a program diverges, i.e., for all $H$ and $n$, $\cdot; s \rightarrow^n H; \text{skip}$ cannot be derived

Example: $\text{while } 1 \text{ skip}$

By induction on $n$, but requires a stronger induction hypothesis
More General Proofs

We can prove properties of executing all programs (satisfying another property)

Example: If $H$ and $s$ have no negative constants and $H; s \rightarrow^* H'; s'$, then $H'$ and $s'$ have no negative constants.

Example: If for all $H$, we know $s_1$ and $s_2$ terminate, then for all $H$, we know $H;(s_1; s_2)$ terminates.