Language Design

What have we been up to?

- Define a programming language
  - we’ve been fairly formal
  - pretty close to SML if you squint real hard

- Define a type system
  - outlaw bad programs that “get stuck”
  - sound: no typable programs get stuck
  - incomplete: knocked out some OK programs too, oh well
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Elsewhere in the Universe (or the other side of campus)

What do logicians do?
  ▶ Define formal logics
    ▶ tools to precisely state propositions
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  ▶ tools to figure out which propositions are true
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  ▷ tools to precisely state propositions

▷ Define proof systems
  ▷ tools to figure out which propositions are true

Turns out, we did that too!
Punchline

We are accidental logicians!
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The Curry-Howard Isomorphism

- Proofs : Propositions :: Programs : Types
- proofs are to propositions as programs are to types
Woah. Back up a second. Logic?!
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Let’s trim down our (explicitly typed) simply-typed \( \lambda \)-calculus to:

\[
e ::= x \mid \lambda x. \; e \mid ee \\
    \mid (e, e) \mid e.1 \mid e.2 \\
    \mid A(e) \mid B(e) \mid \text{match } e \text{ with } A x. \; e \mid B x. \; e
\]

\[
\tau ::= b \mid \tau \to \tau \mid \tau \ast \tau \mid \tau + \tau
\]

- Lambdas, Pairs, and Sums
- Any number of base types \( b_1, b_2, \ldots \)
- No constants (can add one or more if you want)
- No \text{fix}
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- Any number of base types \( b_1, b_2, \ldots \)
- No constants (can add one or more if you want)
- No fix

What good is this?!

Well, even sans constants, plenty of terms type-check with \( \Gamma = \cdot \).
$\lambda x:b. \ x$ has type
\[ \lambda x : b. \ x \]

has type

\[ b \rightarrow b \]
\[ \lambda x : b_1. \ \lambda f : b_1 \rightarrow b_2. \ f \ x \]

has type
\[ \lambda x : b_1. \ \lambda f : b_1 \rightarrow b_2. \ f \ x \]

has type

\[ b_1 \rightarrow (b_1 \rightarrow b_2) \rightarrow b_2 \]
\[
\lambda x : b_1 \to b_2 \to b_3. \ \lambda y : b_2. \ \lambda z : b_1. \ x \ z \ y
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\[ (b_1 \to b_2 \to b_3) \to b_2 \to b_1 \to b_3 \]
\[ \lambda x : b_1. \ (A(x), A(x)) \]

has type
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\[ b_1 \rightarrow ((b_1 + b_7) \ast (b_1 + b_4)) \]
\[ \lambda f : b_1 \rightarrow b_3. \lambda g : b_2 \rightarrow b_3. \lambda z : b_1 + b_2. \]
\[(\text{match } z \text{ with } Ax. f x \mid Bx. g x)\]

has type
\[
\lambda f : b_1 \to b_3. \ \lambda g : b_2 \to b_3. \ \lambda z : b_1 + b_2. \\
\text{(match } z \text{ with } A x. \ f \ x \ | \ B x. \ g \ x) \\
\]

has type

\[
(b_1 \to b_3) \to (b_2 \to b_3) \to (b_1 + b_2) \to b_3
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\[ \lambda x: b_1 \ast b_2. \lambda y: b_3. ((y, x.1), x.2) \]

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\[ (b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2) \]
Empty and Nonempty Types

Just saw a few “nonempty” types

- $\tau$ nonempy if closed term $e$ has type $\tau$
- $\tau$ empty otherwise
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Are there any empty types?
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What does this one mean?

\[ b_1 + (b_1 \to b_2) \]
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Ohwell, now for a totally irrelevant tangent!
Totally irrelevant tangent.
Propositional Logic

Suppose we have some set \( b_1, b_2, \ldots \)

- e.g. “ML is better than Haskell”

Then, using standard operators \( \supset, \land, \lor \), we can define formulas:

- e.g. “ML is better than Haskell” \( \land \) “Haskell is not pure”

Some formulas are tautologies: by virtue of their structure, they are always true regardless of the truth of their constituent propositions.

- e.g. \( p_1 \supset p_1 \)

Not too hard to build a proof system to establish tautologyhood.
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Suppose we have some set $b$ of basic propositions $b_1, b_2, \ldots$

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p ::= b | p \supset p | p \land p | p \lor p
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Proof System

\[ \Gamma ::= \cdot \mid \Gamma, p \]
Proof System

\[\Gamma ::= \cdot | \Gamma, p\]

\[\Gamma \vdash p\]

\[
\frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2}
\]
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\[ \Gamma ::= \cdot \mid \Gamma, p \]

\[ \Gamma \vdash p \]

\[ \frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2} \quad \frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_1} \]
Proof System

\[ \Gamma ::= \cdot | \Gamma, p \]

\[ \Gamma \vdash p \]

\[ \begin{array}{c}
\frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2}
\end{array} \]

\[ \begin{array}{c}
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\end{array} \quad \begin{array}{c}
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\end{array} \]
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\[ \frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \lor p_2} \]
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\[ \frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \lor p_2} \quad \frac{\Gamma \vdash p_2}{\Gamma \vdash p_1 \lor p_2} \]
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\( \Gamma ::= \cdot \mid \Gamma, p \)

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\[ \frac{\Gamma \vdash p_1 \lor p_2}{\Gamma \vdash p_1} \]

\[ \frac{\Gamma \vdash p_1 \lor p_2}{\Gamma \vdash p_2} \]

\[ \frac{\Gamma \vdash p_1 \lor p_2 \quad \Gamma, p_1 \vdash p_3 \quad \Gamma, p_2 \vdash p_3}{\Gamma \vdash p_3} \]
Proof System

\[ \Gamma ::= \cdot | \Gamma, p \]

\[ \Gamma \vdash p \]

\[ \frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2} \]

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\[ \frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \lor p_2} \quad \frac{\Gamma \vdash p_2}{\Gamma \vdash p_1 \lor p_2} \]

\[ \frac{\Gamma \vdash p_1 \lor p_2 \quad \Gamma, p_1 \vdash p_3 \quad \Gamma, p_2 \vdash p_3}{\Gamma \vdash p_3} \]

\[ p \in \Gamma \quad \Gamma \vdash p \]
Proof System

\[ \Gamma ::= \cdot \mid \Gamma, p \]

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\[ \frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2} \quad \frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_1} \quad \frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_2} \]

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\[ \frac{\Gamma \vdash p_1 \lor p_2 \quad \Gamma, p_1 \vdash p_3 \quad \Gamma, p_2 \vdash p_3}{\Gamma \vdash p_3} \]

\[ p \in \Gamma \quad \frac{\Gamma, p_1 \vdash p_2}{\Gamma \vdash p_1 \supset p_2} \]
**Proof System**

\[ \Gamma ::= \cdot | \Gamma, p \]

- \( \Gamma \vdash p \)
- \( \frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2} \)
- \( \frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_1} \)
- \( \frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_2} \)
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- \( \frac{p \in \Gamma}{\Gamma \vdash p} \)
- \( \frac{\Gamma, p_1 \vdash p_2}{\Gamma \vdash p_1 \lor p_2} \)
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- \( \frac{\Gamma \vdash p_1 \lor p_2}{\Gamma \vdash p_2} \)
Wait a second...
Wait a second…
Wait a second... ZOMG!

That's exactly our type system! Just erase terms, change each $\tau$ to a $p$, and translate $\to$ to $\supset$, $*$ to $\land$, $+$ to $\lor$.

\[
\Gamma \vdash e : \tau
\]

\[
\begin{align*}
\Gamma \vdash e_1 : \tau_1 & \quad \Gamma \vdash e_2 : \tau_2 & \quad \Gamma \vdash e : \tau_1 \ast \tau_2 & \quad \Gamma \vdash e : \tau_1 \ast \tau_2 \\
\Gamma \vdash (e_1, e_2) : \tau_1 \ast \tau_2 & \quad \Gamma \vdash e.1 : \tau_1 & \quad \Gamma \vdash e.2 : \tau_2
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau_1 & \quad \Gamma \vdash e : \tau_2 \\
\Gamma \vdash A(e) : \tau_1 + \tau_2 & \quad \Gamma \vdash B(e) : \tau_1 + \tau_2
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau_1 + \tau_2 & \quad \Gamma, x:\tau_1 \vdash e_1 : \tau & \quad \Gamma, y:\tau_2 \vdash e_2 : \tau \\
\Gamma \vdash \text{match } e \text{ with } Ax. \ e_1 \mid By. \ e_2 : \tau
\end{align*}
\]

\[
\begin{align*}
\Gamma(x) = \tau & \quad \Gamma, x : \tau_1 \vdash e : \tau_2 \\
\Gamma \vdash x : \tau & \quad \Gamma \vdash \lambda x. \ e : \tau_1 \to \tau_2 \\
\Gamma \vdash e_1 : \tau_2 \to \tau_1 & \quad \Gamma \vdash e_2 : \tau_2 \\
\Gamma \vdash e_1 \ e_2 : \tau_1
\end{align*}
\]
What does it all mean? The Curry-Howard Isomorphism.

- Given a well-typed closed term, take the typing derivation, erase the terms, and have a propositional-logic proof

- Given a propositional-logic proof, there exists a closed term with that type

- A term that type-checks is a proof — it tells you exactly how to derive the logical formula corresponding to its type
What does it all mean? The Curry-Howard Isomorphism.

- Given a well-typed closed term, take the typing derivation, erase the terms, and have a propositional-logic proof.

- Given a propositional-logic proof, there exists a closed term with that type.

- A term that type-checks is a *proof* — it tells you exactly how to derive the logical formula corresponding to its type.

- Constructive (hold that thought) propositional logic and simply-typed lambda-calculus with pairs and sums are *the same thing*.
  - Computation and logic are *deeply* connected.
  - $\lambda$ is no more or less made up than implication.

- Revisit our examples under the logical interpretation...
\[ \lambda x : b. \ x \]

is a proof that

\[ b \rightarrow b \]
\[ \lambda x : b_1. \lambda f : b_1 \to b_2. f \ x \]

is a proof that

\[ b_1 \to (b_1 \to b_2) \to b_2 \]
\[ \lambda x : b_1 \rightarrow b_2 \rightarrow b_3. \lambda y : b_2. \lambda z : b_1. x \, z \, y \]

is a proof that

\[ (b_1 \rightarrow b_2 \rightarrow b_3) \rightarrow b_2 \rightarrow b_1 \rightarrow b_3 \]
\[
\lambda x : b_1 \cdot (A(x), A(x))
\]

is a proof that

\[
b_1 \rightarrow ((b_1 + b_7) \ast (b_1 + b_4))
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\[ \lambda f : b_1 \rightarrow b_3. \ \lambda g : b_2 \rightarrow b_3. \ \lambda z : b_1 + b_2. \]

\[
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\[
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\[ \lambda x : b_1 \ast b_2. \ \lambda y : b_3. \ ((y, x.1), x.2) \]

is a proof that

\[ (b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2) \]
So what?

Because:

▶ This is just fascinating (glad I’m not a dog)
▶ Don’t think of logic and computing as distinct fields
▶ Thinking “the other way” can help you know what’s possible/impossible
▶ Can form the basis for theorem provers
▶ Type systems should not be *ad hoc* piles of rules!
So what?

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- Type systems should not be *ad hoc* piles of rules!

So, every typed $\lambda$-calculus is a proof system for some logic...

Is STLC with pairs and sums a *complete* proof system for propositional logic? Almost...
Classical vs. Constructive

Classical propositional logic has the “law of the excluded middle”:

\[ \Gamma \vdash p_1 + (p_1 \rightarrow p_2) \]

(Think “\( p + \neg p \)” – also equivalent to double-negation \( \neg \neg p \rightarrow p \) )
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STLC does not support this law; for example, no closed expression has type \( b_1 + (b_1 \rightarrow b_2) \)
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Logics without this rule are called constructive. They’re useful because proofs “know how the world is” and “are executable” and “produce examples”
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Can still “branch on possibilities” by making the excluded middle an explicit assumption:

\[( (p_1 + (p_1 \to p_2)) * (p_1 \to p_3) * ((p_1 \to p_2) \to p_3) ) \to p_3 \]
Classical vs. Constructive, an Example

Theorem: There exist irrational numbers $a$ and $b$ such that $a^b$ is rational.

Classical Proof: Let $x = \sqrt{2}$. Either $x$ is rational or it is irrational. If $x$ is rational, let $a = b = \sqrt{2}$, done. If $x$ is irrational, let $a = x$ and $b = x$. Since $(\sqrt{2} \cdot \sqrt{2}) = \sqrt{2}^2 = 2$, done.

Well, I guess we know there are some $a$ and $b$ satisfying the theorem... but which ones?

Constructive Proof: Let $a = \sqrt{2}$, $b = \log_{10} 9$. Since $\sqrt{2} \cdot \log_{10} 9 = 9 \cdot \log_{10} \sqrt{2} = 9 \cdot \log_{10} (2^{0.5}) = 9 \cdot 0.5 = 3$, done.

To prove that something exists, we actually had to produce it. SWEET.
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If $x^x$ is irrational, let $a = x^x$ and $b = x$. Since

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2,$$ done.

Well, I guess we know there are some $a$ and $b$ satisfying the theorem... but which ones? LAME.

Constructive Proof:

Let $a = \sqrt{2}$, $b = \log_2 9$.

Since $\sqrt{2}^{\log_2 9} = 9^{\log_2 \sqrt{2}} = 9^{\log_2 (2^{0.5})} = 9^{0.5} = 3$, done.

To prove that something exists, we actually had to produce it. SWEET.
Classical vs. Constructive, a Perspective

Constructive logic allows us to distinguish between things that classical logic lumps together.
Classical vs. Constructive, a Perspective

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Consider “$P$ is true.” vs. “It would be absurd if $P$ were false.”

$P$ vs. $\neg \neg P$
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Those are different things, but classical logic can’t tell.
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Our friends Gödel and Gentzen gave us this nice result:

$P$ is provable in classical logic iff $\neg\neg P$ is provable in constructive logic.
Fix

A “non-terminating proof” is no proof at all.

Remember the typing rule for fix:

\[
\Gamma \vdash e : \tau \rightarrow \tau \\
\Gamma \vdash \text{fix } e : \tau
\]

That let’s us prove anything! Example: \text{fix } \lambda x:b. x has type \( b \)

So the “logic” is inconsistent (and therefore worthless)

Related: In ML, a value of type ’a never terminates normally (raises an exception, infinite loop, etc.)

```ml
let rec f x = f x
let z = f 0
```
Last word on Curry-Howard

It’s not just STLC and constructive propositional logic

Every logic has a corresponding typed λ calculus (and no consistent logic has something as “powerful” as fix).
Last word on Curry-Howard

It’s not just STLC and constructive propositional logic

Every logic has a corresponding typed $\lambda$ calculus (and no consistent logic has something as “powerful” as fix).

If you remember one thing: the typing rule for function application is modus ponens