Being Less Restrictive

“Will a \( \lambda \) term get stuck?” is undecidable, so a sound, decidable type system can always be made less restrictive.

An “uninteresting” rule that is sound but not “admissible”:

\[
\frac{\Gamma \vdash e_1 : \tau}{\Gamma \vdash \text{if } \text{true} \ e_1 \ e_2 : \tau}
\]

We’ll study ways to give one term many types (“polymorphism”).

Fact: The version of STLC with explicit argument types \((\lambda x : \tau. \ e)\) has no polymorphism:
If \(\Gamma \vdash e : \tau_1\) and \(\Gamma \vdash e : \tau_2\), then \(\tau_1 = \tau_2\).

Fact: Even without explicit types, many “reuse patterns” do not type-check. Example:
\[
(\lambda f. (f \ 0, f \ \text{true})) (\lambda x. (x, x))
\]
(evaluation to \(((0, 0), (\text{true}, \text{true}))\)).

Today

This lecture is about subtyping.

- Let more terms type-check without adding any new operational behavior
  - But at end consider coercions

- Continue using STLC as our core model

- Complementary to type variables which we will do later
  - Parametric polymorphism \((\forall)\), a.k.a. generics
  - First-class ADTs \((\exists)\)

- Even later: OOP, dynamic dispatch, inheritance vs. subtyping

Motto: Subtyping is not a matter of opinion!
Records

We’ll use records to motivate subtyping:

\[
\begin{align*}
\text{e} & \quad ::= \quad \ldots \quad | \quad \{ l_1 = e_1, \ldots, l_n = e_n \} \quad | \quad e.l \\
\text{\tau} & \quad ::= \quad \ldots \quad | \quad \{ l_1 : \tau_1, \ldots, l_n : \tau_n \} \\
\text{v} & \quad ::= \quad \ldots \quad | \quad \{ l_1 = v_1, \ldots, l_n = v_n \}
\end{align*}
\]

\[
\begin{align*}
\{ l_1 = v_1, \ldots, l_n = v_n \}. l_i & \rightarrow v_i
\end{align*}
\]

\[
\begin{align*}
e_i \rightarrow e_i' & \quad \quad e \rightarrow e' \\
\{ l_1 = v_1, \ldots, l_{i-1} = v_{i-1}, l_i = e_i, \ldots, l_n = e_n \} & \rightarrow \{ l_1 = v_1, \ldots, l_{i-1} = v_{i-1}, l_i = e_i', \ldots, l_n = e_n \}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e_1 : \tau_1 & \quad \ldots \quad \Gamma \vdash e_n : \tau_n \quad \text{labels distinct} \\
\Gamma & \vdash \{ l_1 = e_1, \ldots, l_n = e_n \} : \{ l_1 : \tau_1, \ldots, l_n : \tau_n \}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \{ l_1 : \tau_1, \ldots, l_n : \tau_n \} & \quad 1 \leq i \leq n \\
\Gamma \vdash e.l_i : \tau_i
\end{align*}
\]

Should this typecheck?

\[
(\lambda x : \{ l_1 : \text{int}, l_2 : \text{int} \}. \ x.l_1 + x.l_2) \{ l_1 = 3, l_2 = 4, l_3 = 5 \}
\]

Right now, it doesn’t, but it won’t get stuck

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\]

Right now, it doesn’t, but it won’t get stuck

Suggests width subtyping:

\[
\tau_1 \leq \tau_2
\]

\[
\{ l_1 : \tau_1, \ldots, l_n : \tau_n, l : \tau \} \leq \{ l_1 : \tau_1, \ldots, l_n : \tau_n \}
\]

And one one new type-checking rule: Subsumption

\[
\begin{align*}
\text{SUBSUMPTION} \\
\Gamma \vdash e : \tau' & \quad \tau' \leq \tau \\
\Gamma \vdash e : \tau
\end{align*}
\]
Permutation

Does this program type-check? Does it get stuck?

\[ (\lambda x: \{l_1: \text{int}, l_2: \text{int}\}. x.l_1 + x.l_2)\{l_2=3; l_1=4\} \]

Suggests permutation subtyping:

\[ \{l_1: \tau_1, \ldots, l_i-1: \tau_{i-1}, l_i: \tau_i, \ldots, l_n: \tau_n\} \leq \{l_1: \tau_1, \ldots, l_i: \tau_i, l_i-1: \tau_{i-1}, \ldots, l_n: \tau_n\} \]

Example with width and permutation: Show

\[ \vdash \{l_1=7, l_2=8, l_3=9\} : \{l_2: \text{int}, l_1: \text{int}\} \]
Permutation

Does this program type-check? Does it get stuck?

\[(\lambda x: \{l_1: \text{int}, l_2: \text{int}\}. x.l_1 + x.l_2)\{l_2=3; l_1=4\}\]

Suggests permutation subtyping:

\[\{l_1: \tau_1, \ldots, l_{i-1}: \tau_{i-1}, l_i: \tau_i, \ldots, l_n: \tau_n\} \leq \{l_1: \tau_1, \ldots, l_{i-1}: \tau_{i-1}, l_i: \tau_i, \ldots, l_n: \tau_n\}\]

Example with width and permutation: Show

\[\Gamma \vdash \{l_1=7, l_2=8, l_3=9\} : \{l_2: \text{int}, l_1: \text{int}\}\]

It’s no longer clear there is an (efficient, sound, complete) type-checking algorithm

- They sometimes exist and sometimes don’t
- Here they do

Transitivity

Subtyping is always transitive, so add a rule for that:

\[\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3 \quad \Rightarrow \quad \tau_1 \leq \tau_3\]

Or just use the subsumption rule multiple times. Or both.

In any case, type-checking is no longer syntax-directed: There may be 0, 1, or many different derivations of \(\Gamma \vdash e : \tau\)

- And also potentially many ways to show \(\tau_1 \leq \tau_2\)

Hopefully we could define an algorithm and prove it “answers yes” if and only if there exists a derivation

Digression: Efficiency

With our semantics, width and permutation subtyping make perfect sense

But it would be nice to compile \(e.l\) down to:

1. evaluate \(e\) to a record stored at an address \(a\)
2. load \(a\) into a register \(r_1\)
3. load field \(l\) from a fixed offset (e.g., 4) into \(r_2\)

Many type systems are engineered to make this easy for compiler writers

Makes restrictions seem odd if you do not know techniques for implementing high-level languages

Named types can avoid this, but make code less flexible

Digression continued

With width subtyping alone, the strategy is easy

With permutation subtyping alone, it’s easy but have to “alphabetize”

With both, it’s not easy...

\[\begin{align*}
f_1 : \{l_1 : \text{int}\} &\rightarrow \text{int} \\
f_2 : \{l_2 : \text{int}\} &\rightarrow \text{int} \\
x_1 = \{l_1 = 0, l_2 = 0\} &\quad x_2 = \{l_2 = 0, l_3 = 0\} \\
f_1(x_1) &\quad f_2(x_1) \\
f_1(x_2) &\quad f_2(x_2)
\end{align*}\]

Can use dictionary-passing (look up offset at run-time) and maybe optimize away (some) lookups
So far

- A new subtyping judgement and a new typing rule
  subsumption
- Width, permutation, and transitivity

\[ \tau_1 \leq \tau_2 \quad \frac{l_1:\tau_1, \ldots, l_n:\tau_n, l:\tau}{\{l_1:\tau_1, \ldots, l_n:\tau_n\}} \]

\[ \frac{\{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\}} {\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3 \quad \tau_1 \leq \tau_3} \]

\[ \frac{\Gamma \vdash e : \tau'} {\Gamma \vdash e : \tau} \quad \frac{\tau' \leq \tau} {\Gamma \vdash e : \tau} \]

Now: This is all much more useful if we extend subtyping so it can be used on “parts” of larger types:
- Example: Can’t yet use subsumption on a record field’s type
- Example: There are no supertypes yet of \( \tau_1 \rightarrow \tau_2 \)

Depth

Does this program type-check? Does it get stuck?

\( (\lambda x:\{l_1:\{l_3:\text{int}\}, l_2:\text{int}\}. x.l_1.l_3 + x.l_2)\{l_1=\{l_3=3, l_4=9\}, l_2=4\} \)

Suggests depth subtyping

\[ \tau_i \leq \tau'_i \quad \frac{\{l_1:\tau_1, \ldots, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau'_i, \ldots, l_n:\tau_n\}} {\{l_1:\tau_1, \ldots, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau'_i, \ldots, l_n:\tau_n\}} \]

(With permutation subtyping, can just have depth on left-most field)

Soundness of this rule depends crucially on fields being immutable!
- Depth subtyping is unsound in the presence of mutation
- Trade-off between power (mutation) and sound expressiveness (depth subtyping)
- Homework 4 explores mutation and subtyping
Function subtyping

Given our rich subtyping on records (and/or other primitives), how do we extend it to other types, notably \( \tau_1 \rightarrow \tau_2 \)?

For example, we’d like \( \text{int} \rightarrow \{l_1: \text{int}, l_2: \text{int}\} \leq \text{int} \rightarrow \{l_1: \text{int}\} \) so we can pass a function of the subtype somewhere expecting a function of the supertype.

\[
\frac{\tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4}{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4}
\]

Also want: \( \tau \leq \tau \)

Example: \( \lambda x : \{l_1: \text{int}, l_2: \text{int}\}. \{l_1 = x.l_2, l_2 = x.l_1\} \)

can have type \( \{l_1: \text{int}, l_2: \text{int}, l_3: \text{int}\} \rightarrow \{l_1: \text{int}\} \)

but not \( \{l_1: \text{int}\} \rightarrow \{l_1: \text{int}\} \)

Jargon: Function types are \textit{contravariant} in their argument and \textit{covariant} in their result.

- Depth subtyping means immutable records are covariant in their fields.
Function subtyping, cont'd

\[
\frac{\tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4}{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4}
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but not \( \{l_1: \text{int}\} \rightarrow \{l_1: \text{int}\} \)

Jargon: Function types are contravariant in their argument and covariant in their result

- Depth subtyping means immutable records are covariant in their fields

This is unintuitive enough that you, a friend, or a manager, will some day be convinced that functions can be covariant in their arguments. THIS IS ALWAYS WRONG (UNSOUN). Remember (?) that a PL professor JUMPED UP AND DOWN about this.

Maintaining soundness

Our Preservation and Progress Lemmas still “work” in the presence of subsumption

- So in theory, any subtyping mistakes would be caught when trying to prove soundness!

In fact, it seems too easy: induction on typing derivations makes the subsumption case easy:

- Progress: One new case if typing derivation \( \vdash e : \tau \) ends with subsumption. Then \( \vdash e : \tau' \) via a shorter derivation, so by induction a value or takes a step.

- Preservation: One new case if typing derivation \( \vdash e : \tau \) ends with subsumption. Then \( \vdash e : \tau' \) via a shorter derivation, so by induction if \( e \rightarrow e' \) then \( \vdash e' : \tau' \). So use subsumption to derive \( \vdash e' : \tau \).

Hmm...

Summary of subtyping rules

\[
\frac{\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3}{\tau_1 \leq \tau_3 \quad \tau \leq \tau}
\]

\[
\begin{align*}
\{l_1: \tau_1, \ldots, l_n: \tau_n\} &\leq \{l_1: \tau_1, \ldots, l_n: \tau_n\} \\
\{l_1: \tau_1, \ldots, l_i: \tau_i, \ldots, l_n: \tau_n\} &\leq \\
\{l_1: \tau_1, \ldots, l_i: \tau_i', \ldots, l_n: \tau_n\}
\end{align*}
\]

Notes:

- As always, elegantly handles arbitrarily large syntax (types)
- For other types, e.g., sums or pairs, would have more rules, deciding carefully about co/contravariance of each position

Ah, Canonical Forms

That’s because Canonical Forms is where the action is:

- If \( \vdash v : \{l_1: \tau_1, \ldots, l_n: \tau_n\} \), then \( v \) is a record with fields \( l_1, \ldots, l_n \)
- If \( \vdash v : \tau_1 \rightarrow \tau_2 \), then \( v \) is a function

We need these for the “interesting” cases of Progress

Now have to use induction on the typing derivation (may end with many subsumptions) and induction on the subtyping derivation (e.g., “going up the derivation” only adds fields)

- Canonical Forms is typically trivial without subtyping; now it requires some work

Note: Without subtyping, Preservation is a little “cleaner” via induction on \( e \rightarrow e' \), but with subtyping it’s much cleaner via induction on the typing derivation

- That’s why we did it that way
A matter of opinion?

If subsumption makes well-typed terms get stuck, it is wrong

We might allow less subsumption (e.g., for efficiency), but we shall not allow more than is sound

But we have been discussing “subset semantics” in which $e : \tau$ and $\tau \leq \tau'$ means $e$ is a $\tau'$

- There are “fewer” values of type $\tau$ than of type $\tau'$, but not really

Very tempting to go beyond this, but you must be very careful...

But first we need to emphasize a really nice property of our current setup: *Types never affect run-time behavior*

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**Coercion Semantics**

Wouldn’t it be great if...

- $\text{int} \leq \text{float}$
- $\text{int} \leq \{l_1: \text{int}\}$
- $\tau \leq \text{string}$
- we could “overload the cast operator”

For these proposed $\tau \leq \tau'$ relationships, we need a run-time action to turn a $\tau$ into a $\tau'$

- Called a coercion

Could use float_of_int and similar but programmers whine about it

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**Erasure**

A program type-checks or does not. If it does, it evaluates just like in the untyped $\lambda$-calculus. More formally, we have:

1. Our language with types (e.g., $\lambda x : \tau. \; e$, $A_{\tau_1+\tau_2}(e)$, etc.) and a semantics

2. Our language without types (e.g., $\lambda x. \; e$, $A(e)$, etc.) and a different (but very similar) semantics

3. An *erasure* metafunction from first language to second

4. An equivalence theorem: Erasure commutes with evaluation

This useful (for reasoning and efficiency) fact will be less obvious (but true) with parametric polymorphism

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**Implementing Coercions**

If coercion $C$ (e.g., float_of_int) “witnesses” $\tau \leq \tau'$ (e.g., int $\leq$ float), then we insert $C$ where $\tau$ is subsumed to $\tau'$

So translation to the untyped language depends on where subsumption is used. So it’s from *typing derivations* to programs.
Implementing Coercions

If coercion $C$ (e.g., `float_of_int`) "witnesses" $\tau \leq \tau'$ (e.g., `int \leq float`), then we insert $C$ where $\tau$ is subsumed to $\tau'$

So translation to the untyped language depends on where subsumption is used. So it’s from *typing derivations* to programs.

But typing derivations aren’t unique: uh-oh

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Coherence

Coercions need to be *coherent*, meaning they don’t have these problems

More formally, programs are deterministic even though type checking is not—any typing derivation for $e$ translates to an equivalent program

Alternately, can make (complicated) rules about where subsumption occurs and which subtyping rules take precedence

- Hard to understand, remember, implement correctly

It’s a mess...

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C++

Semi-Example: Multiple inheritance a la C++

```cpp
class C2 {};  
class C3 {};  
class C1 : public C2, public C3 {};  
class D {  
    public:  
        int f(class C2) { return 0; }  
        int f(class C3) { return 1; }  
    };  
int main() { return D().f(C1()); }  
```

Note: A compile-time error "ambiguous call"

Note: Same in Java with interfaces (“reference is ambiguous”)

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Implementing Coercions

If coercion $C$ (e.g., `float_of_int`) "witnesses" $\tau \leq \tau'$ (e.g., `int \leq float`), then we insert $C$ where $\tau$ is subsumed to $\tau'$

So translation to the untyped language depends on where subsumption is used. So it’s from *typing derivations* to programs.

But typing derivations aren’t unique: uh-oh

Example 1:
- Suppose `int \leq float` and $\tau \leq string$
- Consider $\vdash print.string(34) : unit$

Example 2:
- Suppose `int \leq \{l_1: int\}`
- Consider $34 == 34$, where `==` is equality on ints or pointers
Upcasts and Downcasts

- "Subset" subtyping allows "upcasts"
- "Coercive subtyping" allows casts with run-time effect
- What about "downcasts"?

That is, should we have something like:

\[
\text{if\_hastype}(\tau, e_1) \text{ then } x. e_2 \text{ else } e_3
\]

Roughly, if at run-time \( e_1 \) has type \( \tau \) (or a subtype), then bind it to \( x \) and evaluate \( e_2 \). Else evaluate \( e_3 \). Avoids having exceptions.

- Not hard to formalize

Downcasts

Can't deny downcasts exist, but here are some bad things about them:

- Types don't erase – you need to represent \( \tau \) and \( e_1 \)'s type at run-time. (Hidden data fields)
- Breaks abstractions: Before, passing \( \{ l_1 = 3, l_2 = 4 \} \) to a function taking \( \{ l_1 : \text{int} \} \) hid the \( l_2 \) field, so you know it doesn't change or affect the callee

Some better alternatives:

- Use ML-style datatypes — the programmer decides which data should have tags
- Use parametric polymorphism — the right way to do container types (not downcasting results)