**Review**

\[
\begin{align*}
\text{e} &::= \lambda x.e \mid x \mid e\ e \mid c \\
\text{v} &::= \lambda x.e \\
\tau &::= \text{int} \mid \tau \rightarrow \tau \\
\Gamma &::= \cdot \mid \Gamma, x : \tau
\end{align*}
\]

\[
\begin{array}{c}
\frac{\Gamma \vdash c : \text{int}}{\Gamma \vdash x : \Gamma(x)} \quad \frac{\Gamma \vdash \lambda x.e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2} \quad \frac{}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau}
\end{array}
\]

Preservation: If \( \cdot \vdash e : \tau \) and \( e \rightarrow e' \), then \( \cdot \vdash e' : \tau \).
Progress: If \( \cdot \vdash e : \tau \), then \( e \) is a value or \( \exists e' \) such that \( e \rightarrow e' \).

---

**Adding Stuff**

Time to use STLC as a foundation for understanding other
common language constructs

We will add things via a *principled methodology* thanks to a *proper education*

- Extend the syntax
- Extend the operational semantics
  - Derived forms (syntactic sugar), or
  - Direct semantics
- Extend the type system
- Extend soundness proof (new stuck states, proof cases)

In fact, extensions that add new types have even more structure
Derived forms

- let seems just like λ, so can make it a derived form
  - let \( x = e_1 \) in \( e_2 \) “a macro” / “desugars to” \((λx. e_2) e_1\)
  - A “derived form”

(Harder if \( λ \) needs explicit type)

Or just define the semantics to replace let with \( λ \):

\[
\text{let } x = e_1 \text{ in } e_2 \rightarrow (λx. e_2) e_1
\]

These 3 semantics are different in the state-sequence sense
\((e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_n)\)

- But (totally) equivalent and you could prove it (not hard)

Note: ML type-checks let and \( λ \) differently (later topic)
Note: Don’t desugar early if it hurts error messages!

Booleans and Conditionals

\[
e ::= \ldots | \text{true} | \text{false} | \text{if } e_1 e_2 e_3
\]

\[
v ::= \ldots | \text{true} | \text{false}
\]

\[
τ ::= \ldots | \text{bool}
\]

\[
\begin{align*}
e \rightarrow e' \\
(\text{if } e_1 \rightarrow e'_1) (e_2, e_2) \rightarrow (e'_1, e_2)
\end{align*}
\]

\[
\begin{align*}
e \rightarrow e' \\
(\text{if } e_2 \rightarrow e'_2) (v_1, e_2) \rightarrow (v_1, e'_2)
\end{align*}
\]

\[
\begin{align*}
e \rightarrow e' \\
(\text{if } e_3 \rightarrow e'_3) e.1 \rightarrow e'.1
\end{align*}
\]

\[
\begin{align*}
e \rightarrow e' \\
(\text{if } e_1 \rightarrow e'_1) e.2 \rightarrow e'.2
\end{align*}
\]

Pairs (CBV, left-right)

- \( \text{let } x = e_1 \text{ in } e_2 \) \(\rightarrow (λx. e_2) e_1\)
- \( A \ “\text{derived form”} \)

These 3 semantics are different in the state-sequence sense
\((e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_n)\)

- But (totally) equivalent and you could prove it (not hard)

Note: ML type-checks let and \( λ \) differently (later topic)
Note: Don’t desugar early if it hurts error messages!

Pairs continued

- \( \text{if } e_1 \rightarrow e'_1 \)
- \( \text{if } e_2 \rightarrow e'_2 \)
- \( \text{if } e_3 \rightarrow e'_3 \)

- Will learn more concise notation later (evaluation contexts)

Pairs (CBV, left-right)

\[
\begin{align*}
e \equiv \ldots & | (e, e) | e.1 | e.2 \\
v \equiv \ldots & | (v, v) \\
τ \equiv \ldots & | τ * τ
\end{align*}
\]

\[
\begin{align*}
(\text{if } e_1 \rightarrow e'_1) (e_2, e_2) \rightarrow (e'_1, e_2)
\end{align*}
\]

\[
\begin{align*}
(\text{if } e_2 \rightarrow e'_2) (v_1, e_2) \rightarrow (v_1, e'_2)
\end{align*}
\]

\[
\begin{align*}
(\text{if } e_1 \rightarrow e'_1) e.1 \rightarrow e'.1
\end{align*}
\]

\[
\begin{align*}
(\text{if } e_2 \rightarrow e'_2) e.2 \rightarrow e'.2
\end{align*}
\]

Small-step can be a pain

- Large-step needs only 3 rules
- Will learn more concise notation later (evaluation contexts)

Canonical Forms: If \( \vdash v : τ_1 * τ_2 \), then \( v \) has the form \((v_1, v_2)\)

Progress: New cases using Canonical Forms are \( v.1 \) and \( v.2 \)

Preservation: For primitive reductions, inversion gives the result directly
Records

Records are like \( n \)-ary tuples except with named fields

- Field names are not variables; they do not \( \alpha \)-convert

\[
e \ ::= \ldots \mid \{l_1 = e_1; \ldots; l_n = e_n\} \mid e.l
\]

\[
v \ ::= \ldots \mid \{l_1 = v_1; \ldots; l_n = v_n\}
\]

\[
\tau \ ::= \ldots \mid \{l_1 : \tau_1; \ldots; l_n : \tau_n\}
\]

\[
e_i \rightarrow e'_i
\]

\[
\{l_1=v_1, \ldots, l_{i-1}=v_{i-1}, l_i=e_i, \ldots, l_n=e_n\} \rightarrow \{l_1=v_1, \ldots, l_{i-1}=v_{i-1}, l_i=e'_i, \ldots, l_n=e_n\}
\]

\[
1 \leq i \leq n
\]

\[
\Gamma \vdash e_1 : \tau_1 \ldots \Gamma \vdash e_n : \tau_n \quad \text{labels distinct}
\]

\[
\Gamma \vdash \{l_1 = e_1, \ldots, l_n = e_n\} : \{l_1 : \tau_1, \ldots, l_n : \tau_n\}
\]

\[
\Gamma \vdash e : \{l_1 : \tau_1, \ldots, l_n : \tau_n\} \quad 1 \leq i \leq n
\]

\[
\Gamma \vdash e.l_i : \tau_i
\]

Sums

What about ML-style datatypes:

\[
\text{type } t = A \mid B \text{ of int} \mid C \text{ of int * t}
\]

1. Tagged variants (i.e., discriminated unions)

2. Recursive types

3. Type constructors (e.g., type 'a mylist = \ldots)

4. Named types

For now, just model (1) with (anonymous) sum types

- (2) is in a later lecture, (3) is straightforward, and (4) we’ll discuss informally

\[
e \ ::= \ldots \mid A(e) \mid B(e) \mid \text{match } e \text{ with } A x. e \mid B x. e
\]

\[
v \ ::= \ldots \mid A(v) \mid B(v)
\]

\[
\tau \ ::= \ldots \mid \tau_1 + \tau_2
\]

- Only two constructors: \( A \) and \( B \)

- All values of any sum type built from these constructors

- So \( A(e) \) can have any sum type allowed by \( e \)'s type

- No need to declare sum types in advance

- Like functions, will “guess the type” in our rules
Sums operational semantics

\[
\begin{align*}
\text{match } A(v) \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 & \rightarrow e_1[v/x] \\
\text{match } B(v) \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 & \rightarrow e_2[v/y] \\
\end{align*}
\]

\[
\begin{array}{c}
e \rightarrow e' \\
A(e) \rightarrow A(e') \\
B(e) \rightarrow B(e')
\end{array}
\]

\[
\text{match } e \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 \rightarrow \text{match } e' \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2
\]

\[
\text{match} \text{ has binding occurrences, just like pattern-matching}
\]

(Definition of substitution must avoid capture, just like functions)

What is going on

Feel free to think about \textit{tagged values} in your head:

\begin{itemize}
\item A tagged value is a pair of:
  \begin{itemize}
  \item A tag A or B (or 0 or 1 if you prefer)
  \item The (underlying) value
  \end{itemize}
\item A match:
  \begin{itemize}
  \item Checks the tag
  \item Binds the variable to the (underlying) value
  \end{itemize}
\end{itemize}

This much is just like OCaml and related to homework 2

Sums Typing Rules

Inference version (not trivial to infer; can require annotations)

\[
\begin{array}{c}
\Gamma \vdash e : \tau_1 \\
\Gamma \vdash A(e) : \tau_1 + \tau_2 \\
\Gamma \vdash B(e) : \tau_1 + \tau_2 \\
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash e : \tau_1 + \tau_2 \\
\Gamma, x : \tau_1 \vdash e_1 : \tau \\
\Gamma, y : \tau_2 \vdash e_2 : \tau \\
\end{array}
\]

\[
\Gamma \vdash \text{match } e \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 : \tau
\]

Key ideas:

\begin{itemize}
\item For constructor-uses, “other side can be anything”
\item For \texttt{match}, both sides need same type
  \begin{itemize}
  \item Don’t know which branch will be taken, just like an \texttt{if}.
  \item In fact, can drop explicit booleans and encode with sums:
    E.g., \texttt{bool = int + int, true = A(0), false = B(0)}
  \end{itemize}
\end{itemize}

Sums Type Safety

Canonical Forms: If \( \cdot \vdash v : \tau_1 + \tau_2 \), then there exists a \( v_1 \) such that either \( v \) is \( A(v_1) \) and \( \cdot \vdash v_1 : \tau_1 \) or \( v \) is \( B(v_1) \) and \( \cdot \vdash v_1 : \tau_2 \)

\begin{itemize}
\item Progress for \texttt{match} \( v \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 \) follows, as usual, from Canonical Forms
\item Preservation for \texttt{match} \( v \text{ with } Ax. \ e_1 \ \mid \ By. \ e_2 \) follows from the type of the underlying value and the Substitution Lemma
\item The Substitution Lemma has new “hard” cases because we have new binding occurrences
\item But that’s all there is to it (plus lots of induction)
\end{itemize}
What are sums for?

- Pairs, structs, records, aggregates are fundamental data-builders
- Sums are just as fundamental: “this or that not both”
- You have seen how OCaml does sums (datatypes)
- Worth showing how C and Java do the same thing
  - A primitive in one language is an idiom in another

Sums in C

```c
type t = A of t1 | B of t2 | C of t3
match e with A x -> ...
```

One way in C:

```c
struct t {
    enum {A, B, C} tag;
    union {t1 a; t2 b; t3 c;} data;
};
... switch(e->tag){ case A: t1 x=e->data.a; ...}
```

- No static checking that tag is obeyed
- As fat as the fattest variant (avoidable with casts)
- Mutation costs us again!

Sums in Java

```java
type t = A of t1 | B of t2 | C of t3
match e with A x -> ...
```

One way in Java (`t4` is the match-expression’s type):

```java
abstract class t {abstract t4 m();}
class A extends t { t1 x; t4 m(){...}}
class B extends t { t2 x; t4 m(){...}}
class C extends t { t3 x; t4 m(){...}}
... e.m() ...
```

- A new method in `t` and subclasses for each match expression
- Supports extensibility via new variants (subclasses) instead of extensibility via new operations (match expressions)

Pairs vs. Sums

You need both in your language

- With only pairs, you clumsily use dummy values, waste space, and rely on unchecked tagging conventions
- Example: replace `int + (int -> int)` with `int * (int * (int -> int))`

Pairs and sums are “logical duals” (more on that later)

- To make a `τ1 * τ2` you need a `τ1 and a τ2`
- To make a `τ1 + τ2` you need a `τ1 or a τ2`
- Given a `τ1 * τ2`, you can get a `τ1 or a τ2` (or both; your "choice")
- Given a `τ1 + τ2`, you must be prepared for either a `τ1 or τ2` (the value’s "choice")
Base Types and Primitives, in general

What about floats, strings, ...?
Could add them all or do something more general...

Parameterize our language/semantics by a collection of base types \((b_1, \ldots, b_n)\) and primitives \((p_1 : \tau_1, \ldots, p_n : \tau_n)\). Examples:
- `concat` : `string → string → string`
- `toInt` : `float → int`
- “hello” : `string`

For each primitive, assume if applied to values of the right types it produces a value of the right type.

Together the types and assumed steps tell us how to type-check and evaluate \(p_i \, v_1 \ldots v_n\) where \(p_i\) is a primitive.

We can prove soundness once and for all given the assumptions.

Recursion

We won’t prove it, but every extension so far preserves termination.

A Turing-complete language needs some sort of loop, but our lambda-calculus encoding won’t type-check, nor will any encoding of equal expressive power.

So instead add an explicit construct for recursion.
You might be thinking `let rec` \( f \, x = e \), but we will do something more concise and general but less intuitive.

Using `fix`

To use `fix` like `let rec`, just pass it a two-argument function where the first argument is for recursion.

- Not shown: `fix` and tuples can also encode mutual recursion.

Example:
\[
(\text{fix } \lambda f. \lambda n. \text{if } (n<1) 1 (n * (f(n-1)))) 5
\]
Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

Example:

\[(\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1))))\ 5\]
\[\rightarrow\]
\[(\lambda n. \text{ if } (n<1) 1 (n \times ((\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1)))))(n - 1))))\ 5\]

Using fix

To use fix like let rec, just pass it a two-argument function where the first argument is for recursion

Example:

\[(\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1))))\ 5\]
\[\rightarrow\]
\[(\lambda n. \text{ if } (n<1) 1 (n \times ((\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1)))))(n - 1))))\ 5\]
\[\rightarrow\]
\[\text{if } (5<1) 1 (5 \times ((\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1)))))(5 - 1))\]
\[\rightarrow^2\]
\[5 \times ((\text{fix } \lambda f. \lambda n. \text{ if } (n<1) 1 (n \times (f(n - 1)))))(5 - 1)\]
Why called fix?

In math, a fix-point of a function $g$ is an $x$ such that $g(x) = x$

- This makes sense only if $g$ has type $\tau \to \tau$ for some $\tau$
- A particular $g$ could have have 0, 1, 39, or infinity fix-points
- Examples for functions of type $\text{int} \to \text{int}$:
  - $\lambda x. x + 1$ has no fix-points
  - $\lambda x. x \times 0$ has one fix-point
  - $\lambda x. \text{absolute_value}(x)$ has an infinite number of fix-points
  - $\lambda x. \text{if } (x < 10 \&\& x > 0) x 0$ has 10 fix-points

Higher types

At higher types like $(\text{int} \to \text{int}) \to (\text{int} \to \text{int})$, the notion of fix-point is exactly the same (but harder to think about)

- For what inputs $f$ of type $\text{int} \to \text{int}$ is $g(f) = f$

Examples:

- $\lambda f. \lambda x. (f x) + 1$ has no fix-points
- $\lambda f. \lambda x. (f x) \times 0$ (or just $\lambda f. \lambda x. 0$) has 1 fix-point
  - The function that always returns 0
  - In math, there is exactly one such function (cf. equivalence)
- $\lambda f. \lambda x. \text{absolute_value}(f x)$ has an infinite number of fix-points: Any function that never returns a negative result

Back to factorial

Now, what are the fix-points of $\lambda f. \lambda x. \text{if } (x < 1) 1 (x \times (f(x-1)))$?

It turns out there is exactly one (in math): the factorial function!

And $\text{fix } \lambda f. \lambda x. \text{if } (x < 1) 1 (x \times (f(x-1)))$ behaves just like the factorial function

- That is, it behaves just like the fix-point of $\lambda f. \lambda x. \text{if } (x < 1) 1 (x \times (f(x-1)))$
- In general, $\text{fix}$ takes a function-taking-function and returns its fix-point

Typing fix

$$
\Gamma \vdash e : \tau \to \tau \\
\Gamma \vdash \text{fix } e : \tau
$$

Math explanation: If $e$ is a function from $\tau$ to $\tau$, then $\text{fix } e$, the fixed-point of $e$, is some $\tau$ with the fixed-point property

- So it’s something with type $\tau$

Operational explanation: $\text{fix } \lambda x. e'$ becomes $e'[\text{fix } \lambda x. e' / x]$

- The substitution means $x$ and $\text{fix } \lambda x. e'$ need the same type
- The result means $e'$ and $\text{fix } \lambda x. e'$ need the same type

Note: The $\tau$ in the typing rule is usually insantiated with a function type

- e.g., $\tau_1 \to \tau_2$, so $e$ has type $(\tau_1 \to \tau_2) \to (\tau_1 \to \tau_2)$

Note: Proving soundness is straightforward!
General approach

We added let, booleans, pairs, records, sums, and fix

- **let** was syntactic sugar
- **fix** made us Turing-complete by “baking in” self-application
- The others *added types*

Whenever we add a new form of type \( \tau \) there are:

- Introduction forms (ways to make values of type \( \tau \))
- Elimination forms (ways to use values of type \( \tau \))

What are these forms for functions? Pairs? Sums?

When you add a new type, think “what are the intro and elim forms”?

Termination

Surprising fact: If \( \cdot \vdash e : \tau \) in STLC with all our additions except **fix**, then there exists a \( v \) such that \( e \rightarrow^* v \)

- That is, all programs terminate

So termination is trivially decidable (the constant ”yes” function), so our language is not Turing-complete

The proof requires more advanced techniques than we have learned so far because the size of expressions and typing derivations does not decrease with each program step

- Could present it in about an hour if desired

Non-proof:

- Recursion in \( \lambda \) calculus requires some sort of self-application
- Easy fact: For all \( \Gamma, x, \) and \( \tau \), we cannot derive \( \Gamma \vdash x : \tau \)

Anonymity

We added many forms of types, all *unnamed* a.k.a. *structural*. Many real PLs have (all or mostly) *named* types:

- Java, C, C++: all record types (or similar) have names
  - Omitting them just means compiler makes up a name
- OCaml sum types and record types have names

A never-ending debate:

- Structural types allow more code reuse: good
- Named types allow less code reuse: good
- Structural types allow generic type-based code: good
- Named types let type-based code distinguish names: good

The theory is often easier and simpler with structural types