We are Language Designers!

What have we done?
- Define a programming language
  - we were fairly formal
  - still pretty close to OCaml if you squint real hard
- Define a type system
  - outlaw *bad programs* that "get stuck"
  - sound: no typable programs get stuck
  - incomplete: knocked out some OK programs too, ohwell

Elsewhere in the Universe (or the other side of campus)

What do logicians do?
- Define formal logics
  - tools to precisely state propositions
- Define proof systems
  - tools to figure out which propositions are true

Turns out, we did that too!

Punchline

We are accidental logicians!

The Curry-Howard Isomorphism
- Proofs : Propositions :: Programs : Types
- proofs are to propositions as programs are to types
Woah. Back up a second. Logic?!

Let’s trim down our (explicitly typed) simply-typed λ-calculus to:

\[
e ::= x | \lambda x. e | e \ e \\
    | (e, e) | e.1 | e.2 \\
    | A(e) | B(e) | \text{match } e \text{ with } A. e | B. e
\]

\[
\tau ::= b | \tau \to \tau | \tau * \tau | \tau + \tau
\]

- Lambdas, Pairs, and Sums
- Any number of base types \(b_1, b_2, \ldots\)
- No constants (can add one or more if you want)
- No fix

What good is this?!

Well, even sans constants, plenty of terms type-check with \(\Gamma = \cdot\)

\[
\lambda x: b. \ x \\
\lambda x: b_1. \lambda f: b_1 \to b_2. \ f \ x
\]

has type

\[b \to b\]

has type

\[b_1 \to (b_1 \to b_2) \to b_2\]
\[ \lambda x : b_1 \to b_2 \to b_3, \lambda y : b_2, \lambda z : b_1. \ x \ z \ y \]

has type

\[ (b_1 \to b_2 \to b_3) \to b_2 \to b_1 \to b_3 \]
Empty and Nonempty Types

Just saw a few “nonempty” types

- $\tau$ nonempty if closed term $e$ has type $\tau$
- $\tau$ empty otherwise

Are there any empty types?

Sure!

$I_1 \to I_2 \to \left( I_2 \to I_1 \right) \to I_2$

What does this one mean?

$I_1 + \left( I_1 \to I_2 \right)$

I wonder if there’s any way to distinguish empty vs. nonempty...

Ohwell, now for a totally irrelevant tangent!

Propositional Logic

Suppose we have some set $b$ of basic propositions $b_1, b_2, \ldots$

- e.g. “ML is better than Haskell”

Then, using standard operators $\lor, \land, \lor$, we can define formulas:

$p ::= b \mid p \lor p \mid p \land p \mid p \lor p$

- e.g. “ML is better than Haskell” $\land$ “Haskell is not pure”

Some formulas are tautologies: by virtue of their structure, they are always true regardless of the truth of their constituent propositions.

- e.g. $p_1 \supset p_1$

Not too hard to build a proof system to establish tautologyhood.

Proof System

$$
\Gamma ::= \cdot \mid \Gamma, p
$$

$$
\Gamma \vdash p
$$

$$
\frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1 \land p_2}
\quad
\frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_1}
\quad
\frac{\Gamma \vdash p_1 \land p_2}{\Gamma \vdash p_2}
\quad
\frac{\Gamma \vdash p_1}{\Gamma \vdash p_1 \lor p_2}
\quad
\frac{\Gamma \vdash p_2}{\Gamma \vdash p_1 \lor p_2}
\quad
\frac{\Gamma \vdash p_1 \lor p_2 \quad \Gamma, p_1 \vdash p_3 \quad \Gamma, p_2 \vdash p_3}{\Gamma \vdash p_3}
$$

$$
p \in \Gamma
\quad
\frac{\Gamma \vdash p}{\Gamma \vdash p_1 \lor p_2}
\quad
\frac{\Gamma \vdash p_1 \lor p_2}{\Gamma \vdash p_2}
\quad
\frac{\Gamma \vdash p_1 \quad \Gamma \vdash p_2}{\Gamma \vdash p_1}$$. 

Totally irrelevant tangent.
Wait a second... ZOMG!

That’s exactly our type system! Just erase terms, change each \( \tau \) to a \( p \), and translate \( \to \) to \( \supset \), \( * \) to \( \land \), \( + \) to \( \lor \).

\[
\begin{align*}
\Gamma & \vdash e : \tau \\
\Gamma & \vdash e_1 : \tau_1 \quad \Gamma & \vdash e_2 : \tau_2 \\
\Gamma & \vdash (e_1, e_2) : \tau_1 * \tau_2 \\
\Gamma & \vdash e : \tau_1 \quad \Gamma & \vdash e_2 : \tau_2 \\
\Gamma & \vdash \lambda x.e : \tau_1 \to \tau_2 \\
\Gamma & \vdash e_1 : \tau_2 \to \tau_1 \quad \Gamma & \vdash e_2 : \tau_2
\end{align*}
\]

What does it all mean? The Curry-Howard Isomorphism.

- Given a well-typed closed term, take the typing derivation, erase the terms, and have a propositional-logic proof

\[ \lambda x : b. \ x \]

is a proof that \( b \to b \)

- Given a propositional-logic proof, there exists a closed term with that type

- A term that type-checks is a proof — it tells you exactly how to derive the logic formula corresponding to its type

- Constructive (hold that thought) propositional logic and simply-typed lambda-calculus with pairs and sums are the same thing.
  - Computation and logic are deeply connected
  - \( \lambda \) is no more or less made up than implication

- Revisit our examples under the logical interpretation...
\[ \lambda x : b_1. \lambda f : b_1 \to b_2. \, f \, x \quad \text{is a proof that} \quad b_1 \to (b_1 \to b_2) \to b_2 \]

\[ \lambda x : b_1 \to b_2 \to b_3. \lambda y : b_2. \lambda z : b_1. \, x \, z \, y \quad \text{is a proof that} \quad (b_1 \to b_2 \to b_3) \to b_2 \to b_1 \to b_3 \]

\[ \lambda x : b_1. \, (A(x), A(x)) \quad \text{is a proof that} \quad b_1 \to ((b_1 + b_7) \ast (b_1 + b_4)) \]

\[ \lambda f : b_1 \to b_3. \lambda g : b_2 \to b_3. \lambda z : b_1 + b_2. \, (\text{match } z \text{ with } A x. \, f \, x \mid B x. \, g \, x) \quad \text{is a proof that} \quad (b_1 \to b_3) \to (b_2 \to b_3) \to (b_1 + b_2) \to b_3 \]
$\lambda x : b_1 \ast b_2. \lambda y : b_3. ((y, x.1), x.2)$

is a proof that

$(b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2)$

So what?
Because:

- This is just fascinating (glad I’m not a dog)
- Don’t think of logic and computing as distinct fields
- Thinking “the other way” can help you know what’s possible/impossible
- Can form the basis for theorem provers
- Type systems should not be *ad hoc* piles of rules!

So, every typed $\lambda$-calculus is a proof system for some logic...

Is STLC with pairs and sums a complete proof system for propositional logic? Almost...

**Classical vs. Constructive, an Example**

Theorem: There exist irrational numbers $a$ and $b$ such that $a^b$ is rational.

Classical Proof:

Let $x = \sqrt{2}$. Either $x^x$ is rational or it is irrational. 

If $x^x$ is rational, let $a = b = \sqrt{2}$, done.

If $x^x$ is irrational, let $a = x^x$ and $b = x$. Since 

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2,$$

done.

Well, I guess we know there are some $a$ and $b$ satisfying the theorem... but which ones? LAME.

Constructive Proof:

Let $a = \sqrt{2}$, $b = \log_2 9$.

Since $\sqrt{2}^{\log_2 9} = 9^{\log_2 (\sqrt{2}^2)} = 9^{0.5} = 3$, done.

To prove that something exists, we actually had to produce it. SWEET.
Constructive logic allows us to distinguish between things that classical logic just crudely lumps together.

Consider “\(P\) is true.” vs. “It would be absurd if \(P\) were false.”

- \(P\) vs. \(\neg\neg P\)

Those are different things, but classical logic is too clumsy to tell.

Our friends Gödel and Gentzen gave us this nice result:

\[ P \text{ is provable in classical logic iff } \neg\neg P \text{ is provable in constructive logic.}\]

A “non-terminating proof” is no proof at all.

Remember the typing rule for \textbf{fix}:

\[
\frac{\Gamma \vdash e : \tau \to \tau}{\Gamma \vdash \text{fix } e : \tau}
\]

That lets us prove anything! Example: \textbf{fix} \(\lambda x : b.\ x\) has type \(b\)

So the “logic” is \textit{inconsistent} (and therefore worthless)

Related: In ML, a value of type \(\forall a. a\) never terminates normally (raises an exception, infinite loop, etc.)

\[
\text{let rec f x = f x}\\
\text{let z = f 0}
\]

It’s not just STLC and constructive propositional logic

\textit{Every} logic has a corresponding typed \(\lambda\) calculus (and no consistent logic has something as “powerful” as \textbf{fix}).

- Example: When we add universal types (“generics”) in a later lecture, that corresponds to adding universal quantification

If you remember one thing: the typing rule for function application is \textit{modus ponens}