CSE505: Graduate Programming Languages

Lecture 6 — Little Trusted Languages; Equivalence

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Looking back, looking forward

This is the last lecture using IMP (hooray!). Done:

- Abstract syntax
- Operational semantics (large-step and small-step)
- Semantic properties of (sets of) programs
- “Pseudo-denotational” semantics

Now:

- Packet-filter languages and other examples
- Equivalence of programs in a semantics
- Equivalence of different semantics

Next lecture: Local variables, lambda-calculus
Packet Filters

A very simple view of packet filters:

- Some bits come in off the wire
- Some application(s) want the “packet” and some do not (e.g., port number)
- For safety, only the O/S can access the wire
- For extensibility, the applications accept/reject packets

Conventional solution goes to user-space for every packet and app that wants (any) packets

Faster solution: Run app-written filters in kernel-space
What we need

Now the O/S writer is defining the packet-filter language!

Properties we wish of (untrusted) filters:

1. Don’t corrupt kernel data structures
2. Terminate (within a time bound)
3. Run fast (the whole point)

Should we download some C/assembly code? (Get 1 of 3)

Should we make up a language and “hope” it has these properties?
Language-based approaches

1. Interpret a language
   + clean operational semantics, + portable, - may be slow (+ filter-specific optimizations), - unusual interface

2. Translate a language into C/assembly
   + clean denotational semantics, + employ existing optimizers, - upfront cost, - unusual interface

3. Require a conservative subset of C/assembly
   + normal interface, - too conservative w/o help

IMP has taught us about (1) and (2) — we’ll get to (3)
A General Pattern

Packet filters move the code to the data rather than data to the code

General reasons: performance, security, other?

Other examples:
- Query languages
- Active networks
- Client-side web scripts (Javascript)
Equivalence motivation

- Program equivalence (we change the program):
  - code optimizer
  - code maintainer

- Semantics equivalence (we change the language):
  - interpreter optimizer
  - language designer
    - (prove properties for equivalent semantics with easier proof)

Note: Proofs may seem easy with the right semantics and lemmas
  - (almost never start off with right semantics and lemmas)

Note: Small-step operational semantics often has harder proofs, but models more interesting things
What is equivalence?

Equivalence depends on what is observable!
What is equivalence?

Equivalence depends on what is observable!

- Partial I/O equivalence (if terminates, same ans)

- Total I/O equivalence (same termination behavior, same ans)

- Total heap equivalence (same termination behavior, same heaps)

- All (almost all?) variables have the same value

- Equivalence plus complexity bounds

- Is $O(2^n)$ really equivalent to $O(n)$?

- Is “runs within 10ms of each other” important?

- Syntactic equivalence (perhaps with renaming)

- Too strict to be interesting?

In PL, equivalence most often means total I/O equivalence
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  - `while 1 skip` equivalent to everything

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- Equivalence plus complexity bounds
  - Is $O(2^{n^n})$ really equivalent to $O(n)$?
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What is equivalence?

Equivalence depends on what is observable!

- Partial I/O equivalence (if terminates, same ans)
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▶ Equivalence plus complexity bounds
  ▶ Is $O(2^n)$ really equivalent to $O(n)$?
  ▶ Is “runs within 10ms of each other” important?

▶ Syntactic equivalence (perhaps with renaming)
  ▶ Too strict to be interesting?

In PL, equivalence most often means total I/O equivalence
Program Example: Strength Reduction

Motivation: Strength reduction
  ▶ A common compiler optimization due to architecture issues

Theorem: $H; e \ast 2 \Downarrow c$ if and only if $H; e + e \Downarrow c$

Proof sketch:
Program Example: Strength Reduction

Motivation: Strength reduction
  ▶ A common compiler optimization due to architecture issues

Theorem: $H ; e \ast 2 \downarrow c$ if and only if $H ; e + e \downarrow c$

Proof sketch:
  ▶ Prove separately for each direction
Program Example: Strength Reduction

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Theorem: $H; e \ast 2 \Downarrow c$ if and only if $H; e + e \Downarrow c$

Proof sketch:
- Prove separately for each direction
- Invert the assumed derivation, use hypotheses plus a little math to derive what we need
Program Example: Strength Reduction

Motivation: Strength reduction

▶ A common compiler optimization due to architecture issues

Theorem: $H; e * 2 \downarrow c$ if and only if $H; e + e \downarrow c$

Proof sketch:

▶ Prove separately for each direction

▶ Invert the assumed derivation, use hypotheses plus a little math to derive what we need

▶ Hmm, doesn’t use induction. That’s because this theorem isn’t very useful...
Theorem: If $e'$ has a subexpression of the form $e \ast 2$, then $H ; e' \downarrow c'$ if and only if $H ; e'' \downarrow c'$ where $e''$ is $e'$ with $e \ast 2$ replaced with $e + e$.
Program Example: Nested Strength Reduction

Theorem: If $e'$ has a subexpression of the form $e \times 2$, then $H ; e' \downarrow c'$ if and only if $H ; e'' \downarrow c'$ where $e''$ is $e'$ with $e \times 2$ replaced with $e + e$

First some useful metanotation:

\[
C ::= [\cdot] \mid C + e \mid e + C \mid C \times e \mid e \times C
\]

$C[e]$ is “$C$ with $e$ in the hole” (inductive definition of “stapling”)

Crisper statement of theorem:

$H ; C[e \times 2] \downarrow c'$ if and only if $H ; C[e + e] \downarrow c'$
Program Example: Nested Strength Reduction

Theorem: If $e'$ has a subexpression of the form $e \ast 2$, then $H ; e' \Downarrow c'$ if and only if $H ; e'' \Downarrow c'$ where $e''$ is $e'$ with $e \ast 2$ replaced with $e + e$

First some useful metanotation:

$$C ::= [·] | C + e | e + C | C \ast e | e \ast C$$

$C[e]$ is “$C$ with $e$ in the hole” (inductive definition of “stapling”)

Crisper statement of theorem:

$$H ; C[e \ast 2] \Downarrow c'$$ if and only if $$H ; C[e + e] \Downarrow c'$$

Proof sketch: By induction on structure (“syntax height”) of $C$

- The base case ($C = [·]$) follows from our previous proof
- The rest is a long, tedious, (and instructive!) induction
Proof reuse

As we cannot emphasize enough, proving is just like programming

The proof of nested strength reduction had nothing to do with $e \ast 2$ and $e + e$ except in the base case where we used our previous theorem

A much more useful theorem would parameterize over the base case so that we could get the “nested $X$” theorem for any appropriate $X$:

If $(H ; e_1 \downarrow c$ if and only if $H ; e_2 \downarrow c)$,
then $(H ; C[e_1] \downarrow c'$ if and only if $H ; C[e_2] \downarrow c'$)

The proof is identical except the base case is “by assumption”
Small-step program equivalence

These sort of proofs also work with small-step semantics (e.g., our IMP statements), but tend to be more cumbersome, even to state.

Example: The statement-sequence operator is associative. That is,

(a) For all \( n \), if \( H ; s_1; (s_2; s_3) \rightarrow^H H' \); \texttt{skip} then there exist \( H'' \) and \( n' \) such that \( H ; (s_1; s_2); s_3 \rightarrow^{n'} H'' \); \texttt{skip} and \( H''(\texttt{ans}) = H'(\texttt{ans}) \).

(b) If for all \( n \) there exist \( H' \) and \( s' \) such that \( H ; s_1; (s_2; s_3) \rightarrow^n H' \); \( s' \), then for all \( n \) there exist \( H'' \) and \( s'' \) such that \( H ; (s_1; s_2); s_3 \rightarrow^n H'' \); \( s'' \).

(Proof needs a much stronger induction hypothesis.)

One way to avoid it: Prove large-step and small-step semantics equivalent, then prove program equivalences in whichever is easier.
Language Equivalence Example

IMP w/o multiply large-step:

\[
\begin{align*}
\text{CONST} & \quad \text{VAR} \\
H ; c \downarrow c & \quad H ; x \downarrow H(x)
\end{align*}
\]

IMP w/o multiply small-step:

\[
\begin{align*}
\text{SVAR} & \\
H ; x & \rightarrow H(x)
\end{align*}
\]

\[
\begin{align*}
\text{SLEFT} & \\
H ; e_1 & \rightarrow e'_1 \\
H ; e_1 + e_2 & \rightarrow e'_1 + e_2
\end{align*}
\]

\[
\begin{align*}
\text{SADD} & \\
H ; c_1 + c_2 & \rightarrow c_1 + c_2
\end{align*}
\]

Theorem: Semantics are equivalent: \( H ; e \downarrow c \) if and only if \( H ; e \rightarrow^* c \)

Proof: We prove the two directions separately...
Proof, part 1

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H; e \downarrow c$
Proof, part 1

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- **CONST**: Derivation with **CONST** implies $e = c$, and we can derive $H; c \rightarrow^0 c$
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^c c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses $\text{sleft}$ and $\text{sright}$

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- **const**: Derivation with $\text{const}$ implies $e = c$, and we can derive $H; c \rightarrow^0 c$
- **var**: Derivation with $\text{var}$ implies $e = x$ for some $x$ where $H(x) = c$, so derive $H; e \rightarrow^1 c$ with $\text{svar}$
Proof, part 1

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- **CONST**: Derivation with **CONST** implies $e = c$, and we can derive $H; c \rightarrow^0 c$
- **VAR**: Derivation with **VAR** implies $e = x$ for some $x$ where $H(x) = c$, so derive $H; e \rightarrow^1 c$ with **SVAR**
- **ADD**: ...
Part 1, continued

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- ...
- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \downarrow c_1$, and $H; e_2 \downarrow c_2$ for some $e_1, e_2, c_1, c_2$. 
Part 1, continued

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- ... 
- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \downarrow c_1$, and $H; e_2 \downarrow c_2$ for some $e_1, e_2, c_1, c_2$. By induction (twice), $\exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$. 
Part 1, continued

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$
and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- ... 
- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$,
  $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ for some $e_1, e_2, c_1, c_2$.
  By induction (twice), $\exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1$ and
  $H; e_2 \rightarrow^{n_2} c_2$.
  So by our lemma $H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and
  $H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$. 
Part 1, continued

First assume $H : e \downarrow c$ and show $\exists n. H : e \rightarrow^n c$

Lemma (prove it!): If $H : e \rightarrow^n e'$, then $H : e_1 + e \rightarrow^n e_1 + e'$
and $H : e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H : e \downarrow c$

- ... 
- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H : e_1 \downarrow c_1$, and $H : e_2 \downarrow c_2$ for some $e_1, e_2, c_1, c_2$.
By induction (twice), $\exists n_1, n_2. H : e_1 \rightarrow^{n_1} c_1$ and $H : e_2 \rightarrow^{n_2} c_2$.
So by our lemma $H : e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H : c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.
By SADD $H : c_1 + c_2 \rightarrow c_1 + c_2$. 
Part 1, continued

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- ...

- **ADD**: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ for some $e_1, e_2, c_1, c_2$.
  
  By induction (twice), $\exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$.

  So by our lemma $H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.

  By **SADD** $H; c_1 + c_2 \rightarrow c_1 + c_2$.

  So $H; e_1 + e_2 \rightarrow^{n_1 + n_2 + 1} c$. 
Proof, part 2

Now assume \( \exists n. H; e \rightarrow^n c \) and show \( H; e \downarrow c \).
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H ; e \downarrow c$.

Proof by induction on $n$: 

Proof by induction on derivation of $H ; e \rightarrow e'$: 

▶ $svar$: ... 

▶ $sadd$: ... 

▶ $sleft$: ... 

▶ $sright$: ...
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H ; e \downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H ; c \downarrow c$
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H; e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\texttt{CONST}$ lets us derive $H; c \Downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  $\exists e'. \ H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c.$
Proof, part 2

Now assume $\exists n. \; H; e \rightarrow^n c$ and show $H; e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H; c \Downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  $\exists e'. \; H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c$.
  By induction $H; e' \Downarrow c$. 

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Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H \ ; \ e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H \ ; \ c \Downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  $\exists e'. \ H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c$.
  By induction $H \ ; \ e' \Downarrow c$.
  So this lemma suffices: If $H; e \rightarrow e'$ and $H \ ; \ e' \Downarrow c$, then $H \ ; \ e \Downarrow c$. 

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Proof, part 2

Now assume $\exists n. H; e \rightarrow^n c$ and show $H; e \downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H; c \downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  \[ \exists e'. H; e \rightarrow e' \text{ and } H; e' \rightarrow^{n-1} c. \]
  By induction $H; e' \downarrow c$.
  So this lemma suffices: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- $\text{SVAR}$: ...
- $\text{SADD}$: ...
- $\text{SLEFT}$: ...
- $\text{SRIGHT}$: ...
Part 2, key lemma

Lemma: If \( H; e \rightarrow e' \) and \( H; e' \downarrow c \), then \( H; e \downarrow c \).

Prove the lemma by induction on derivation of \( H; e \rightarrow e' \):
Part 2, key lemma

Lemma: If $H; e \to e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \to e'$:

▶ **svar**: Derivation with **svar** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **var**, $H; x \Downarrow H(x)$.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR**: Derivation with **SVAR** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **VAR**, $H; x \Downarrow H(x)$.

- **SADD**: Derivation with **SADD** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **ADD** and two **CONST**, $H; c_1 + c_2 \Downarrow c_1 + c_2$. 
Part 2, key lemma

Lemma: If \( H; e \rightarrow e' \) and \( H; e' \downarrow c \), then \( H; e \downarrow c \).

Prove the lemma by induction on derivation of \( H; e \rightarrow e' \):

- **SVAR**: Derivation with \texttt{SVAR} implies \( e \) is some \( x \) and \( e' = H(x) = c \), so derive, by \texttt{VAR}, \( H; x \downarrow H(x) \).

- **SADD**: Derivation with \texttt{SADD} implies \( e \) is some \( c_1 + c_2 \) and \( e' = c_1 + c_2 = c \), so derive, by \texttt{ADD} and two \texttt{CONST}, \( H; c_1 + c_2 \downarrow c_1 + c_2 \).

- **SLEFT**: Derivation with \texttt{SLEFT} implies \( e = e_1 + e_2 \) and \( e' = e'_1 + e_2 \) and \( H; e_1 \rightarrow e'_1 \) for some \( e_1, e_2, e'_1 \).
Lemma: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR**: Derivation with SVAR implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by VAR, $H; x \downarrow H(x)$.

- **SADD**: Derivation with SADD implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by ADD and two CONST, $H; c_1 + c_2 \downarrow c_1 + c_2$.

- **SLEFT**: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$. Since $e' = e'_1 + e_2$ inverting assumption $H; e' \downarrow c$ gives $H; e'_1 \downarrow c_1, H; e_2 \downarrow c_2$ and $c = c_1 + c_2$. 
Part 2, key lemma

Lemma: If \( H; e \rightarrow e' \) and \( H; e' \downarrow c \), then \( H; e \downarrow c \).

Prove the lemma by induction on derivation of \( H; e \rightarrow e' \):

- **svar**: Derivation with **svar** implies \( e \) is some \( x \) and \( e' = H(x) = c \), so derive, by **var**, \( H; x \downarrow H(x) \).

- **sadd**: Derivation with **sadd** implies \( e \) is some \( c_1 + c_2 \) and \( e' = c_1 + c_2 = c \), so derive, by **add** and two **const**, \( H; c_1 + c_2 \downarrow c_1 + c_2 \).

- **sleft**: Derivation with **sleft** implies \( e = e_1 + e_2 \) and \( e' = e'_1 + e_2 \) and \( H; e_1 \rightarrow e'_1 \) for some \( e_1, e_2, e'_1 \). Since \( e' = e'_1 + e_2 \) inverting assumption \( H; e' \downarrow c \) gives \( H; e'_1 \downarrow c_1, H; e_2 \downarrow c_2 \) and \( c = c_1 + c_2 \). Applying the induction hypothesis to \( H; e_1 \rightarrow e'_1 \) and \( H; e'_1 \downarrow c_1 \) gives \( H; e_1 \downarrow c_1 \).
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **svar**: Derivation with svar implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by var, $H; x \Downarrow H(x)$.

- **sadd**: Derivation with sadd implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by add and two const, $H; c_1 + c_2 \Downarrow c_1 + c_2$.

- **sleft**: Derivation with sleft implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$.

Since $e' = e'_1 + e_2$ inverting assumption $H; e' \Downarrow c$ gives $H; e'_1 \Downarrow c_1, H; e_2 \Downarrow c_2$ and $c = c_1 + c_2$.

Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H; e'_1 \Downarrow c_1$ gives $H; e_1 \Downarrow c_1$.

So use add, $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ to derive $H; e_1 + e_2 \Downarrow c_1 + c_2$.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR**: Derivation with **SVAR** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **VAR**, $H; x \downarrow H(x)$.
- **SADD**: Derivation with **SADD** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **ADD** and two **CONST**, $H; c_1 + c_2 \downarrow c_1 + c_2$.
- **SLEFT**: Derivation with **SLEFT** implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$.
  
  Since $e' = e'_1 + e_2$ inverting assumption $H; e' \downarrow c$ gives $H; e'_1 \downarrow c_1, H; e_2 \downarrow c_2$ and $c = c_1 + c_2$.
  
  Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H; e'_1 \downarrow c_1$ gives $H; e_1 \downarrow c_1$.

  So use **ADD**, $H; e_1 \downarrow c_1$, and $H; e_2 \downarrow c_2$ to derive $H; e_1 + e_2 \downarrow c_1 + c_2$.

- **SRIGHT**: Analogous to **SLEFT**
The cool part, redux

Step through the $\texttt{SLEFT}$ case more visually:

By assumption, we must have derivations that look like this:

$$
\begin{align*}
H; e_1 & \rightarrow e'_1 \\
H; e_1 + e_2 & \rightarrow e'_1 + e_2 \\
H; e'_1 \downarrow c_1 & \\
H; e'_1 + e_2 \downarrow c_1 + c_2
\end{align*}
$$

Grab the hypothesis from the left and the left hypothesis from the right and use induction to get $H; e_1 \downarrow c_1$.

Now go grab the one hypothesis we haven’t used yet and combine it with our inductive result to derive our answer:

$$
\begin{align*}
H; e_1 \downarrow c_1 & \\
H; e_2 \downarrow c_2 \\
H; e_1 + e_2 \downarrow c_1 + c_2
\end{align*}
$$
A nice payoff

Theorem: The small-step semantics is deterministic:

if $H; e \rightarrow^* c_1$ and $H; e \rightarrow^* c_2$, then $c_1 = c_2$
A nice payoff

Theorem: The small-step semantics is deterministic:
if $H; e \rightarrow^* c_1$ and $H; e \rightarrow^* c_2$, then $c_1 = c_2$

Not obvious (see SLEFT and SRIGHT), nor do I know a direct proof

- Given $(((1 + 2) + (3 + 4)) + (5 + 6)) + (7 + 8)$ there are many execution sequences, which all produce 36 but with different intermediate expressions
A nice payoff

Theorem: The small-step semantics is deterministic:
if $H; e \rightarrow^* c_1$ and $H; e \rightarrow^* c_2$, then $c_1 = c_2$

Not obvious (see SLEFT and SRIGHT), nor do I know a direct proof

- Given $((((1 + 2) + (3 + 4)) + (5 + 6)) + (7 + 8))$ there are many execution sequences, which all produce 36 but with different intermediate expressions

Proof:

- Large-step evaluation is deterministic (easy induction proof)
- Small-step and and large-step are equivalent (just proved that)
- So small-step is deterministic
- Convince yourself a deterministic and a nondeterministic semantics can’t be equivalent
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace **WHILE** rule with

\[
\begin{align*}
    & H ; e \downarrow c & c \leq 0 \\
    \hline
    & H ; \text{while } e \ s & \rightarrow H ; \text{skip} \\
\end{align*}
\]

\[
\begin{align*}
    & H ; e \downarrow c & c > 0 \\
    \hline
    & H ; \text{while } e \ s & \rightarrow H ; s ; \text{while } e \ s \\
\end{align*}
\]
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace **WHILE** rule with

\[
\begin{align*}
&\frac{H ; e \Downarrow c \quad c \leq 0}{H ; \text{while } e \text{ s } \rightarrow H ; \text{skip}}
\end{align*}
\]

\[
\begin{align*}
&\frac{H ; e \Downarrow c \quad c > 0}{H ; \text{while } e \text{ s } \rightarrow H \text{ ; s}; \text{while } e \text{ s}}
\end{align*}
\]

Equivalent to our original language
Conclusions

- Equivalence is a subtle concept
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- Some other language-equivalence claims:

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\]

\[
\begin{align*}
H; e \Downarrow c & \quad c > 0 \\
H; \text{while } e s & \rightarrow H; s; \text{while } e s
\end{align*}
\]

Equivalent to our original language

Change syntax of heap and replace **ASSIGN** and **VAR** rules with

\[
\begin{align*}
H; x := e & \rightarrow H, x \mapsto e; \text{skip }
\end{align*}
\]

\[
\begin{align*}
H; H(x) \Downarrow c & \\
H; x \Downarrow c
\end{align*}
\]
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace \texttt{WHILE} rule with

\[
\begin{align*}
H ; e \Downarrow c & \quad c \leq 0 \\
\hline
& H ; \texttt{while } e \texttt{ s } \rightarrow H ; \texttt{skip}
\end{align*}
\]

\[
\begin{align*}
H ; e \Downarrow c & \quad c > 0 \\
\hline
& H ; \texttt{while } e \texttt{ s } \rightarrow H ; s ; \texttt{while } e \texttt{ s}
\end{align*}
\]

Equivalent to our original language

Change syntax of heap and replace \texttt{ASSIGN} and \texttt{VAR} rules with

\[
\begin{align*}
H ; x := e & \rightarrow H, x \mapsto e ; \texttt{skip} \\
\hline
H ; H(x) \Downarrow c & H ; x \Downarrow c
\end{align*}
\]

\textit{NOT} equivalent to our original language