Finally, some formal PL content

For our first *formal language*, let’s leave out functions, objects, records, threads, exceptions, ...

What’s left: integers, mutable variables, control-flow

(Abstract) syntax using a common *metalanguage*:

“A program is a statement \( s \), which is defined as follows”

\[
\begin{align*}
\ s & ::= \ \text{skip} \ | \ x ::= e \ | \ s; s \ | \ \text{if} \ e \ s \ s \ | \ \text{while} \ e \ s \\
\ e & ::= \ c \ | \ x \ | \ e + e \ | \ e \ast e \\
(c & \in \ \{\ldots, -2, -1, 0, 1, 2, \ldots \}) \\
(x & \in \ \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \})
\end{align*}
\]
Syntax Definition

\[ s ::= \text{skip} \mid x ::= e \mid s; s \mid \text{if } e \; s \; s \mid \text{while } e \; s \]
\[ e ::= c \mid x \mid e + e \mid e \ast e \]
\[ (c \in \{..., -2, -1, 0, 1, 2, ... \}) \]
\[ (x \in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \}) \]

- Blue is metanotation: ::= for “can be a” and | for “or”
- Metavariables represent “anything in the syntax class”
- By abstract syntax, we mean that this defines a set of trees
  - Node has some label for “which alternative”
  - Children are more abstract syntax (subtrees) from the appropriate syntax class
Examples

\[ s ::= \text{skip} \mid x := e \mid s; s \mid \text{if} \ e \ s \ s \mid \text{while} \ e \ s \]
\[ e ::= c \mid x \mid e + e \mid e * e \]

Trees:

- Left tree: \( \text{if} \ x \ \text{skip} ; \text{:=} \) \( y 42 x y \)
- Right tree: \( \text{if} \ x \ \text{skip} := x y \) \( y 42 \)
Comparison to ML

```
If(Var("x"),Skip,Seq(Assign("y",Const 42),Assign("x",Var "y")))
Seq(If(Var("x"),Skip,Assign("y",Const 42)),Assign("x",Var "y"))
```

Very similar to trees built with ML datatypes
- ML needs “extra nodes” for, e.g., “e can be a c”
- Also pretending ML’s int is an integer
Comparison to strings

We are used to writing programs in *concrete syntax*, i.e., strings

That can be *ambiguous*: \( \text{if } x \text{ skip } y := 42 ; x := y \)

Since writing strings is such a convenient way to represent trees, we allow ourselves parentheses (or defaults) for disambiguation

- Trees are our “truth” with strings as a “convenient notation”

\( \text{if } x \text{ skip } (y := 42 ; x := y) \) versus \( (\text{if } x \text{ skip } y := 42) ; x := y \)
Last word on concrete syntax

Converting a string into a tree is parsing.

Creating concrete syntax such that parsing is unambiguous is one challenge of grammar design.

- Always trivial if you require enough parentheses or keywords
  - Extreme case: LISP, 1960s; Scheme, 1970s
  - Extreme case: XML, 1990s
- Very well studied in 1970s and 1980s, now typically the least interesting part of a compilers course

For the rest of this course, we start with abstract syntax

- Using strings only as a convenient shorthand and asking if it’s ever unclear what tree we mean
Inductive definition

\[
\begin{align*}
  s &::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \text{ then } s \text{ else } s \mid \text{while } e \text{ do } s \\
  e &::= c \mid x \mid e + e \mid e * e
\end{align*}
\]

This grammar is a finite description of an infinite set of trees.

The apparent self-reference is not a problem, provided the definition uses well-founded induction.

- Just like an always-terminating recursive function uses self-reference but is not a circular definition!

Can give precise meaning to our metanotation & avoid circularity:

- Let \( E_0 = \emptyset \)
- For \( i > 0 \), let \( E_i \) be \( E_{i-1} \) union “expressions of the form \( c, x, e_1 + e_2, \text{ or } e_1 * e_2 \) where \( e_1, e_2 \in E_{i-1} \)”
- Let \( E = \bigcup_{i \geq 0} E_i \)

The set \( E \) is what we mean by our compact metanotation.
Inductive definition

\[
\begin{align*}
  s &::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \text{ s s} \mid \text{while } e \text{ s} \\
  e &::= c \mid x \mid e + e \mid e \ast e
\end{align*}
\]

Let \( E_0 = \emptyset \).

For \( i > 0 \), let \( E_i \) be \( E_{i-1} \) union “expressions of the form \( c \), \( x \), \( e_1 + e_2 \), or \( e_1 \ast e_2 \) where \( e_1, e_2 \in E_{i-1} \)”.

Let \( E = \bigcup_{i \geq 0} E_i \).

The set \( E \) is what we mean by our compact metanotation.

To get it: What set is \( E_1 \)? \( E_2 \)?

Could explain statements the same way: What is \( S_1 \)? \( S_2 \)? \( S \)?
All we have is syntax (sets of abstract-syntax trees), but let’s get the idea of proving things carefully...

Theorem 1: There exist expressions with three constants.
Our First Theorem

There exist expressions with three constants.

Pedantic Proof: Consider $e = 1 + (2 + 3)$. Showing $e \in E_3$ suffices because $E_3 \subseteq E$. Showing $2 + 3 \in E_2$ and $1 \in E_2$ suffices...

PL-style proof: Consider $e = 1 + (2 + 3)$ and definition of $E$.

Theorem 2: All expressions have at least one constant or variable.
Our Second Theorem

All expressions have at least one constant or variable.

Pedantic proof: By induction on $i$, for all $e \in E_i$, $e$ has $\geq 1$ constant or variable.

- **Base:** $i = 0$ implies $E_i = \emptyset$
- **Inductive:** $i > 0$. Consider *arbitrary* $e \in E_i$ by cases:
  - $e \in E_{i-1}$
  - $e = c$
  - $e = x$
  - $e = e_1 + e_2$ where $e_1, e_2 \in E_{i-1}$
  - $e = e_1 * e_2$ where $e_1, e_2 \in E_{i-1}$
A “Better” Proof

All expressions have at least one constant or variable.

PL-style proof: By *structural induction* on (rules for forming an expression) $e$. Cases:

- $c \ldots$
- $x \ldots$
- $e_1 + e_2 \ldots$
- $e_1 \times e_2 \ldots$

Structural induction invokes the induction hypothesis on *smaller* terms. It is equivalent to the pedantic proof, and more convenient in PL.