Finally, some formal PL content

For our first *formal language*, let’s leave out functions, objects, records, threads, exceptions, ...

What’s left: integers, mutable variables, control-flow

(Abstract) syntax using a common *metalanguage*:

“A program is a statement $s$, which is defined as follows”

\[
\begin{align*}
 s & ::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \text{ s } s \mid \text{while } e \text{ s} \\
 e & ::= c \mid x \mid e + e \mid e * e \\
 (c & \in \{ \ldots , -2 , -1 , 0 , 1 , 2 , \ldots \}) \\
 (x & \in \{ x_1 , x_2 , \ldots , y_1 , y_2 , \ldots , z_1 , z_2 , \ldots , \ldots \})
\end{align*}
\]
Syntax Definition

\[
\begin{align*}
  s &::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \text{ s s} \mid \text{while } e \text{ s} \\
  e &::= c \mid x \mid e + e \mid e * e \\
  &\quad (c \in \{\ldots, -2, -1, 0, 1, 2, \ldots \}) \\
  &\quad (x \in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \})
\end{align*}
\]

- Blue is metanotation: ::= for “can be a” and | for “or”

- Metavariables represent “anything in the syntax class”

- By abstract syntax, we mean that this defines a set of trees
  - Node has some label for “which alternative”
  - Children are more abstract syntax (subtrees) from the appropriate syntax class
Examples

\[
\begin{align*}
s & ::= \text{skip} \mid x ::= e \mid s; s \mid \text{if } e s s \mid \text{while } e s \\
e & ::= c \mid x \mid e + e \mid e \ast e
\end{align*}
\]
Comparison to ML

If (Var("x"), Skip, Seq(Assign("y", Const 42), Assign("x", Var "y")))
Seq(If(Var("x"), Skip, Assign("y", Const 42)), Assign("x", Var "y"))

Very similar to trees built with ML datatypes

- ML needs “extra nodes” for, e.g., “e can be a c”
- Also pretending ML’s int is an integer
Comparison to strings

We are used to writing programs in *concrete syntax*, i.e., strings

That can be *ambiguous*: \texttt{if x skip y := 42 ; x := y}

Since writing strings is such a convenient way to represent trees, we allow ourselves parentheses (or defaults) for disambiguation

- Trees are our “truth” with strings as a “convenient notation”

\texttt{if x skip (y := 42 ; x := y) versus (if x skip y := 42) ; x := y}
Last word on concrete syntax

Converting a string into a tree is parsing

Creating concrete syntax such that parsing is unambiguous is one challenge of grammar design

- Always trivial if you require enough parentheses or keywords
  - Extreme case: LISP, 1960s; Scheme, 1970s
  - Extreme case: XML, 1990s
- Very well studied in 1970s and 1980s, now typically the least interesting part of a compilers course

For the rest of this course, we start with abstract syntax

- Using strings only as a convenient shorthand and asking if it’s ever unclear what tree we mean
Inductive definition

\[
\begin{align*}
    s &::= \text{skip} \mid x := e \mid s; s \mid \text{if e s s} \mid \text{while e s} \\
    e &::= c \mid x \mid e + e \mid e * e
\end{align*}
\]

This grammar is a finite description of an infinite set of trees

The apparent self-reference is not a problem, provided the definition uses well-founded induction

- Just like an always-terminating recursive function uses self-reference but is not a circular definition!

Can give precise meaning to our metanotation & avoid circularity:

- Let \( E_0 = \emptyset \)
- For \( i > 0 \), let \( E_i \) be \( E_{i-1} \) union “expressions of the form \( c, x, e_1 + e_2, \) or \( e_1 * e_2 \) where \( e_1, e_2 \in E_{i-1} \)”
- Let \( E = \bigcup_{i \geq 0} E_i \)

The set \( E \) is what we mean by our compact metanotation
Inductive definition

\[ s ::= \text{skip} \mid x ::= e \mid s; s \mid \text{if } e \ s \ s \mid \text{while } e \ s \]
\[ e ::= c \mid x \mid e + e \mid e \ast e \]

- Let \( E_0 = \emptyset \).
- For \( i > 0 \), let \( E_i \) be \( E_{i-1} \) union “expressions of the form \( c, x, e_1 + e_2, \) or \( e_1 \ast e_2 \) where \( e_1, e_2 \in E_{i-1} \)”.
- Let \( E = \bigcup_{i \geq 0} E_i \).

The set \( E \) is what we mean by our compact metanotation.

To get it: What set is \( E_1 \)? \( E_2 \)?
Could explain statements the same way: What is \( S_1 \)? \( S_2 \)? \( S \)?
All we have is syntax (sets of abstract-syntax trees), but let’s get the idea of proving things carefully...

Theorem 1: There exist expressions with three constants.
Our First Theorem

There exist expressions with three constants.

Pedantic Proof: Consider \( e = 1 + (2 + 3) \). Showing \( e \in E_3 \) suffices because \( E_3 \subseteq E \). Showing \( 2 + 3 \in E_2 \) and \( 1 \in E_2 \) suffices...

PL-style proof: Consider \( e = 1 + (2 + 3) \) and definition of \( E \).

Theorem 2: All expressions have at least one constant or variable.
Our Second Theorem

All expressions have at least one constant or variable.

Pedantic proof: By induction on $i$, for all $e \in E_i$, $e$ has $\geq 1$ constant or variable.

- **Base:** $i = 0$ implies $E_i = \emptyset$
- **Inductive:** $i > 0$. Consider arbitrary $e \in E_i$ by cases:
  - $e \in E_{i-1} \ldots$
  - $e = c \ldots$
  - $e = x \ldots$
  - $e = e_1 + e_2$ where $e_1, e_2 \in E_{i-1} \ldots$
  - $e = e_1 \ast e_2$ where $e_1, e_2 \in E_{i-1} \ldots$
A “Better” Proof

All expressions have at least one constant or variable.

PL-style proof: By \textit{structural induction} on (rules for forming an expression) $e$. Cases:

- $c \ldots$
- $x \ldots$
- $e_1 + e_2 \ldots$
- $e_1 \ast e_2 \ldots$

Structural induction invokes the induction hypothesis on \textit{smaller} terms. It is equivalent to the pedantic proof, and more convenient in PL.