Being Less Restrictive

“Will a \( \lambda \) term get stuck?” is undecidable, so a sound, decidable type system can always be made less restrictive

An “uninteresting” rule that is sound but not “admissable”:

\[
\frac{\Gamma \vdash e_1 : \tau}{\Gamma \vdash \text{if true } e_1 \ e_2 : \tau}
\]

We’ll study ways to give one term many types (“polymorphism”)

Fact: The version of STLC with explicit argument types \((\lambda x : \tau. \ e)\) has no polymorphism:

If \(\Gamma \vdash e : \tau_1\) and \(\Gamma \vdash e : \tau_2\), then \(\tau_1 = \tau_2\)

Fact: Even without explicit types, many “reuse patterns” do not type-check. Example: \((\lambda f. \ (f \ 0, f \ \text{true})))(\lambda x. \ (x, x))\)

(evaluates to \(((0, 0), (\text{true, true}))\))
Polymorphism means many things... 

- **Ad hoc polymorphism**: $e_1 + e_2$ in SML$<$C$<$Java$<$C++

- **Ad hoc, cont’d**: Maybe $e_1$ and $e_2$ can have different run-time types and we choose the $+$ based on them

- **Parametric polymorphism**: e.g., $\Gamma \vdash \lambda x. x : \forall \alpha. \alpha \to \alpha$ or with explicit types: $\Gamma \vdash \Lambda \alpha. \lambda x : \alpha. x : \forall \alpha. \alpha \to \alpha$

  (which “compiles” i.e. “erases” to $\lambda x. x$

- **Subtype polymorphism**: `new Vector().add(new C())` is legal Java because `new C()` has types `Object` and `C`

  ...and nothing.

  (More precise terms: “static overloading,” “dynamic dispatch,” “type abstraction,” and “subtyping”)
Today

This lecture is about *subtyping*

- Let more terms type-check without adding any new operational behavior
  - But at end consider *coercions*

- Continue using STLC as our core model

- Complementary to type variables which we will do later
  - Parametric polymorphism ($\forall$), a.k.a. generics
  - First-class ADTs ($\exists$)

- Even later: OOP, dynamic dispatch, inheritance vs. subtyping

Motto: Subtyping is not a matter of opinion!
Records

We’ll use records to motivate subtyping:

\[
e ::= \ldots | \{l_1 = e_1, \ldots, l_n = e_n\} | e.l
\]

\[
\tau ::= \ldots | \{l_1 : \tau_1, \ldots, l_n : \tau_n\}
\]

\[
v ::= \ldots | \{l_1 = v_1, \ldots, l_n = v_n\}
\]

\[
\{l_1 = v_1, \ldots, l_n = v_n\}.l_i \rightarrow v_i
\]

\[
ed_i \rightarrow e_i'
\]

\[
\{l_1 = v_1, \ldots, l_i-1 = v_i-1, l_i = e_i, \ldots, l_n = e_n\} \rightarrow \{l_1 = v_1, \ldots, l_i-1 = v_i-1, l_i = e_i', \ldots, l_n = e_n\}
\]

\[
\Gamma \vdash e_1 : \tau_1 \quad \ldots \quad \Gamma \vdash e_n : \tau_n \quad \text{labels distinct}
\]

\[
\Gamma \vdash \{l_1 = e_1, \ldots, l_n = e_n\} : \{l_1 : \tau_1, \ldots, l_n : \tau_n\}
\]

\[
\Gamma \vdash e : \{l_1 : \tau_1, \ldots, l_n : \tau_n\} \quad 1 \leq i \leq n
\]

\[
\Gamma \vdash e.l_i : \tau_i
\]
Should this typecheck?

\[(\lambda x : \{l_1 : \text{int}, l_2 : \text{int}\}. \ x.l_1 + x.l_2)\{l_1 = 3, l_2 = 4, l_3 = 5\}\]
Should this typecheck?

$$(\lambda x : \{l_1:\text{int}, l_2:\text{int}\}. \ x.l_1 + x.l_2)\{l_1=3, l_2=4, l_3=5\}$$

Right now, it doesn’t, but it won’t get stuck
Should this typecheck?

\((\lambda x : \{l_1:\text{int}, l_2:\text{int}\}. x.l_1 + x.l_2)\{l_1=3, l_2=4, l_3=5\}\)

Right now, it doesn’t, but it won’t get stuck

Suggests width subtyping:

\[ \tau_1 \leq \tau_2 \]

\[ \{l_1:\tau_1, \ldots, l_n:\tau_n, l:\tau\} \leq \{l_1:\tau_1, \ldots, l_n:\tau_n\} \]

And one one new type-checking rule: Subsumption

\[
\text{SUBSUMPTION} \quad \Gamma \vdash e : \tau' \quad \tau' \leq \tau \\
\hline
\Gamma \vdash e : \tau
\]
Now it type-checks

\[ \therefore, x : \{l_1: \text{int}, l_2: \text{int}\} \vdash x \cdot l_1 + x \cdot l_2 : \text{int} \]

\[ \therefore \vdash \lambda x : \{l_1: \text{int}, l_2: \text{int}\}. x \cdot l_1 + x \cdot l_2 : \{l_1: \text{int}, l_2: \text{int}\} \rightarrow \text{int} \]

\[ \therefore \vdash (\lambda x : \{l_1: \text{int}, l_2: \text{int}\}. x \cdot l_1 + x \cdot l_2)\{l_1=3, l_2=4, l_3=5\} : \text{int} \]

Instantiation of Subsumption is highlighted (pardon formatting)

The derivation of the subtyping fact
\[ \{l_1: \text{int}, l_2: \text{int}, l_3: \text{int}\} \leq \{l_1: \text{int}, l_2: \text{int}\} \] would continue, using rules for the \( \tau_1 \leq \tau_2 \) judgment

▶ But here we just use the one axiom we have so far

Clean division of responsibility:
▶ Where to use subsumption
▶ How to show two types are subtypes
Permutation

Does this program type-check? Does it get stuck?

$$(\lambda x: \{l_1: \text{int}, l_2: \text{int}\}. x.l_1 + x.l_2)\{l_2=3; l_1=4\}$$
Permutation

Does this program type-check? Does it get stuck?

\[(\lambda x:\{l_1:\text{int}, l_2:\text{int}\}. \ x.l_1 + x.l_2)\{l_2=3; l_1=4\}\]

Suggests *permutation subtyping*:

\[
\{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \\
\{l_1:\tau_1, \ldots, l_i:\tau_i, l_{i-1}:\tau_{i-1}, \ldots, l_n:\tau_n\}
\]
Permutation

Does this program type-check? Does it get stuck?

\[(\lambda x:\{l_1:\text{int}, l_2:\text{int}\}. \; x.l_1 + x.l_2)\{l_2=3; l_1=4\}\]

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\{l_1:\tau_1, \ldots, l_i:\tau_i, l_{i-1}:\tau_{i-1}, \ldots, l_n:\tau_n\}
\]

Example with width and permutation: Show
\[\vdash \{l_1=7, l_2=8, l_3=9\} : \{l_2:\text{int}, l_1:\text{int}\}\]
Permutation

Does this program type-check? Does it get stuck?

\[(\lambda x:\{l_1:\text{int}, l_2:\text{int}\}. \; x.l_1 + x.l_2)\{l_2=3; l_1=4\}\]

Suggests permutation subtyping:

\[
\{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau_i, l_{i-1}:\tau_{i-1}, \ldots, l_n:\tau_n\}
\]

Example with width and permutation: Show
\[\vdash \{l_1=7, l_2=8, l_3=9\} : \{l_2:\text{int}, l_1:\text{int}\}\]

It’s no longer clear there is an (efficient, sound, complete) type-checking algorithm

➤ They sometimes exist and sometimes don’t

➤ Here they do
Transitivity

Subtyping is always transitive, so add a rule for that:

\[
\frac{\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3}{\tau_1 \leq \tau_3}
\]

Or just use the subsumption rule multiple times. Or both.

In any case, type-checking is no longer syntax-directed: There may be 0, 1, or many different derivations of \(\Gamma \vdash e : \tau\)

- And also potentially many ways to show \(\tau_1 \leq \tau_2\)

Hopefully we could define an algorithm and prove it “answers yes” if and only if there exists a derivation.
Digression: Efficiency

With our semantics, width and permutation subtyping make perfect sense

But it would be nice to compile $e.l$ down to:

1. evaluate $e$ to a record stored at an address $a$
2. load $a$ into a register $r_1$
3. load field $l$ from a fixed offset (e.g., 4) into $r_2$

Many type systems are engineered to make this easy for compiler writers

Makes restrictions seem odd if you do not know techniques for implementing high-level languages
Digression continued

With width subtyping alone, the strategy is easy

With permutation subtyping alone, it’s easy but have to “alphabetize”

With both, it’s not easy...

\[
\begin{align*}
    f_1 : \{l_1 : \text{int}\} &\rightarrow \text{int} & f_2 : \{l_2 : \text{int}\} &\rightarrow \text{int} \\
    x_1 = \{l_1 = 0, l_2 = 0\} & & x_2 = \{l_2 = 0, l_3 = 0\} \\
    f_1(x_1) & & f_2(x_1) & & f_2(x_2)
\end{align*}
\]

Can use dictionary-passing (look up offset at run-time) and maybe optimize away (some) lookups

*Named types* can avoid this, but make code less flexible
So far

- A new *subtyping judgement* and a new typing rule *subsumption*

- Width, permutation, and transitivity

\[ \tau_1 \leq \tau_2 \]

\[ \{l_1:\tau_1, \ldots, l_n:\tau_n, l:\tau\} \leq \{l_1:\tau_1, \ldots, l_n:\tau_n\} \]

\[ \{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau_i, l_{i-1}:\tau_{i-1}, \ldots, l_n:\tau_n\} \]

\[ \frac{\tau_1 \leq \tau_2}{\tau_2 \leq \tau_3} \frac{\tau_1 \leq \tau_3}{\tau_1 \leq \tau_3} \]

Now: This is all much more useful if we extend subtyping so it can be used on “parts” of larger types:

- Example: Can’t yet use subsumption on a record field’s type

- Example: There are no supertypes yet of \( \tau_1 \rightarrow \tau_2 \)
Depth

Does this program type-check? Does it get stuck?

\[(\lambda x: \{l_1: \{l_3: \text{int}\}, l_2: \text{int}\}. x.l_1.l_3 + x.l_2)\{l_1=\{l_3=3, l_4=9\}, l_2=4\}\]
Depth

Does this program type-check? Does it get stuck?

\[(\lambda x:\{l_1:\{l_3:\text{int}\}, l_2:\text{int}\}. x.l_1.l_3 + x.l_2)\{l_1=\{l_3=3, l_4=9\}, l_2=4\}\]

Suggests depth subtyping

\[\tau_i \leq \tau'_i\]

\[
\{l_1:\tau_1, \ldots, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau'_i, \ldots, l_n:\tau_n\}
\]

(With permutation subtyping, can just have depth on left-most field)
Depth

Does this program type-check? Does it get stuck?

\[(\lambda x: \{l_1: \{l_3: \text{int}\}, l_2: \text{int}\}. x.l_1.l_3 + x.l_2)\{l_1=\{l_3=3, l_4=9\}, l_2=4\}\]

Suggests depth subtyping

\[\tau_i \leq \tau'_i\]

\[\{l_1: \tau_1, \ldots, l_i: \tau_i, \ldots, l_n: \tau_n\} \leq \{l_1: \tau_1, \ldots, l_i: \tau'_i, \ldots, l_n: \tau_n\}\]

(With permutation subtyping, can just have depth on left-most field)

Soundness of this rule depends crucially on fields being immutable!

- Depth subtyping is unsound in the presence of mutation
- Trade-off between power (mutation) and sound expressiveness (depth subtyping)
- Homework 4 explores mutation and subtyping
Function subtyping

Given our rich subtyping on records (and/or other primitives), how do we extend it to other types, notably $\tau_1 \rightarrow \tau_2$?

For example, we’d like $\text{int} \rightarrow \{l_1: \text{int}, l_2: \text{int}\} \leq \text{int} \rightarrow \{l_1: \text{int}\}$ so we can pass a function of the subtype somewhere expecting a function of the supertype

$$??$$

$$\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4$$
Function subtyping

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$$
\text{??}
$$

$$
\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4
$$

For a function to have type $\tau_3 \rightarrow \tau_4$ it must return something of type $\tau_4$ (including subtypes) whenever given something of type $\tau_3$ (including subtypes). A function assuming less than $\tau_3$ will do, but not one assuming more. A function returning more than $\tau_4$ but not one returning less.
Function subtyping, cont’d

\[
\frac{\tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4}{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4}
\]

Also want: \( \tau \leq \tau \)

Example: \( \lambda x : \{l_1: \text{int}, l_2: \text{int}\}. \{l_1 = x. l_2, l_2 = x. l_1\} \)
can have type \( \{l_1: \text{int}, l_2: \text{int}, l_3: \text{int}\} \rightarrow \{l_1: \text{int}\} \)
but not \( \{l_1: \text{int}\} \rightarrow \{l_1: \text{int}\} \)
Function subtyping, cont’d

\[ \tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4 \quad \frac{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4}{\tau \leq \tau} \]

Also want: \[ \tau \leq \tau \]

Example: \( \lambda x : \{ l_1 : \text{int}, l_2 : \text{int} \} \cdot \{ l_1 = x \cdot l_2, l_2 = x \cdot l_1 \} \)
can have type \( \{ l_1 : \text{int}, l_2 : \text{int}, l_3 : \text{int} \} \rightarrow \{ l_1 : \text{int} \} \)
but not \( \{ l_1 : \text{int} \} \rightarrow \{ l_1 : \text{int} \} \)

Jargon: Function types are *contravariant* in their argument and *covariant* in their result

- Depth subtyping means immutable records are covariant in their fields
Function subtyping, cont’d

\[
\frac{\tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4}{\tau_1 \to \tau_2 \leq \tau_3 \to \tau_4}
\]

Also want: \( \tau \leq \tau \)

Example: \( \lambda x : \{l_1:int, l_2:int\}. \{l_1 = x.l_2, l_2 = x.l_1\} \)

can have type \( \{l_1:int, l_2:int, l_3:int\} \to \{l_1:int\} \)

but not \( \{l_1:int\} \to \{l_1:int\} \)

Jargon: Function types are *contravariant* in their argument and *covariant* in their result

- Depth subtyping means immutable records are covariant in their fields

This is unintuitive enough that you, a friend, or a manager, will some day be convinced that functions can be covariant in their arguments. **THIS IS ALWAYS WRONG (UN SOUND).**
Summary of subtyping rules

\[
\frac{\tau_1 \leq \tau_2 \quad \tau_2 \leq \tau_3}{\tau_1 \leq \tau_3} \quad \frac{\tau \leq \tau}{\tau \leq \tau}
\]

\[
\{l_1:\tau_1, \ldots, l_n:\tau_n, l:\tau\} \leq \{l_1:\tau_1, \ldots, l_n:\tau_n\}
\]

\[
\{l_1:\tau_1, \ldots, l_{i-1}:\tau_{i-1}, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau_i, l_{i-1}:\tau_{i-1}, \ldots, l_n:\tau_n\}
\]

\[
\frac{\tau_i \leq \tau'_i}{\{l_1:\tau_1, \ldots, l_i:\tau_i, \ldots, l_n:\tau_n\} \leq \{l_1:\tau_1, \ldots, l_i:\tau'_i, \ldots, l_n:\tau_n\}}
\]

\[
\frac{\tau_3 \leq \tau_1 \quad \tau_2 \leq \tau_4}{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4}
\]

Notes:
- As always, elegantly handles arbitrarily large syntax (types)
- For other types, e.g., sums or pairs, would have more rules, deciding carefully about co/contravariance of each position
Maintaining soundness

Our Preservation and Progress Lemmas still “work” in the presence of subsumption

- So in theory, any subtyping mistakes would be caught when trying to prove soundness!

In fact, it seems too easy: induction on typing derivations makes the subsumption case easy:

- Progress: One new case if typing derivation $\cdot \vdash e : \tau$ ends with subsumption. Then $\cdot \vdash e : \tau'$ via a shorter derivation, so by induction a value or takes a step.

- Preservation: One new case if typing derivation $\cdot \vdash e : \tau$ ends with subsumption. Then $\cdot \vdash e : \tau'$ via a shorter derivation, so by induction if $e \rightarrow e'$ then $\cdot \vdash e' : \tau'$. So use subsumption to derive $\cdot \vdash e' : \tau$.

Hmm...
Ah, Canonical Forms

That’s because Canonical Forms is where the action is:

- If \( \cdot \vdash v : \{l_1:\tau_1, \ldots, l_n:\tau_n\} \), then \( v \) is a record with fields \( l_1, \ldots, l_n \)
- If \( \cdot \vdash v : \tau_1 \rightarrow \tau_2 \), then \( v \) is a function

We need these for the “interesting” cases of Progress

Now have to use induction on the typing derivation (may end with many subsumptions) and induction on the subtyping derivation (e.g., “going up the derivation” only adds fields)

- Canonical Forms is typically trivial without subtyping; now it requires some work

Note: Without subtyping, Preservation is a little “cleaner” via induction on \( e \rightarrow e' \), but with subtyping it’s much cleaner via induction on the typing derivation

- That’s why we did it that way
A matter of opinion?

If subsumption makes well-typed terms get stuck, it is \textit{wrong}.

We might allow less subsumption (e.g., for efficiency), but we shall not allow more than is sound.

But we have been discussing “subset semantics” in which $e : \tau$ and $\tau \leq \tau'$ means \textit{e is a} $\tau'$.

- There are “fewer” values of type $\tau$ than of type $\tau'$, but not really.

Very tempting to go beyond this, but you must be very careful...

But first we need to emphasize a really nice property of our current setup: \textit{Types never affect run-time behavior}.
Erasure

A program type-checks or does not. If it does, it evaluates just like in the untyped $\lambda$-calculus. More formally, we have:

1. Our language with types (e.g., $\lambda x : \tau. e$, $A_{\tau_1 + \tau_2}(e)$, etc.) and a semantics

2. Our language without types (e.g., $\lambda x. e$, $A(e)$, etc.) and a different (but very similar) semantics

3. An erasure metafunction from first language to second

4. An equivalence theorem: Erasure commutes with evaluation

This useful (for reasoning and efficiency) fact will be less obvious (but true) with parametric polymorphism
Coercion Semantics

Wouldn’t it be great if...

- \( \text{int} \leq \text{float} \)
- \( \text{int} \leq \{l_1:\text{int}\} \)
- \( \tau \leq \text{string} \)
- we could “overload the cast operator”

For these proposed \( \tau \leq \tau' \) relationships, we need a run-time action to turn a \( \tau \) into a \( \tau' \)

- Called a coercion

Could use \texttt{float_of_int} and similar but programmers whine about it
Implementing Coercions

If coercion $C$ (e.g., `float_of_int`) “witnesses” $\tau \leq \tau'$ (e.g., `int \leq float`), then we insert $C$ where $\tau$ is subsumed to $\tau'$

So translation to the untyped language depends on where subsumption is used. So it’s from *typing derivations* to programs.
Implementing Coercions

If coercion \( C \) (e.g., \texttt{float_of_int}) “witnesses” \( \tau \leq \tau' \) (e.g., \texttt{int} \leq \texttt{float}'), then we insert \( C \) where \( \tau \) is subsumed to \( \tau' \)

So translation to the untyped language depends on where subsumption is used. So it’s from \textit{typing derivations} to programs.

But typing derivations aren’t unique: uh-oh
Implementing Coercions

If coercion $C$ (e.g., `float_of_int`) "witnesses" $\tau \leq \tau'$ (e.g., `int \leq float`), then we insert $C$ where $\tau$ is subsumed to $\tau'$

So translation to the untyped language depends on where subsumption is used. So it’s from typing derivations to programs.

But typing derivations aren’t unique: uh-oh

Example 1:

- Suppose `int \leq float` and $\tau \leq \text{string}$
- Consider $\vdash \text{print_string}(34) : \text{unit}$

Example 2:

- Suppose `int \leq \{l_1:\text{int}\}$
- Consider $34 == 34$, where `==` is equality on ints or pointers
Coherence

Coercions need to be coherent, meaning they don’t have these problems.

More formally, programs are deterministic even though type checking is not—any typing derivation for \( e \) translates to an equivalent program.

Alternately, can make (complicated) rules about where subsumption occurs and which subtyping rules take precedence.

- Hard to understand, remember, implement correctly.

It’s a mess...
Semi-Example: Multiple inheritance a la C++

class C2 {};  
class C3 {};  
class C1 : public C2, public C3 {};  
class D {  
   public: int f(class C2) { return 0; }  
   int f(class C3) { return 1; }  
};  
int main() { return D().f(C1()); }  

Note: A compile-time error “ambiguous call”

Note: Same in Java with interfaces (“reference is ambiguous”)
Upcasts and Downcasts

- “Subset” subtyping allows “upcasts”
- “Coercive subtyping” allows casts with run-time effect
- What about “downcasts”?
Upcasts and Downcasts

- “Subset” subtyping allows “upcasts”
- “Coercive subtyping” allows casts with run-time effect
- What about “downcasts”?

That is, should we have something like:

\[
\text{if\_hastype}(\tau, e_1) \text{ then } x.\ e_2 \text{ else } e_3
\]

Roughly, if at run-time \(e_1\) has type \(\tau\) (or a subtype), then bind it to \(x\) and evaluate \(e_2\). Else evaluate \(e_3\). Avoids having exceptions.

- Not hard to formalize
Downcasts

Can’t deny downcasts exist, but here are some bad things about them:

▶ Types don’t erase – you need to represent $\tau$ and $e_1$’s type at run-time. (Hidden data fields)
▶ Breaks abstractions: Before, passing $\{l_1 = 3, l_2 = 4\}$ to a function taking $\{l_1 : \text{int}\}$ hid the $l_2$ field, so you know it doesn’t change or affect the callee

Some better alternatives:

▶ Use ML-style datatypes — the programmer decides which data should have tags
▶ Use parametric polymorphism — the right way to do container types (not downcasting results)