CSE 505: Concepts of Programming Languages

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Lecture 10—Curry-Howard Isomorphism; Evaluation Contexts
Outline

Two totally different topics:

• Curry-Howard Isomorphism
  – Types are propositions
  – Programs are proofs

• Equivalent ways to express evaluation of $\lambda$-calculus
  – Evaluation contexts
  – Explicit stacks
  – Closures instead of substitution

A series of equivalent implementations from our operational semantics to a fairly efficient “low-level” implementation!

Note: See ML code for much of the second topic

Evaluation contexts / stacks also let us talk about continuations
Curry-Howard Isomorphism

What we did:

- Define a programming language
- Define a type system to rule out programs we don’t want

What logicians do:

- Define a logic (a way to state propositions)
  - Example: Propositional logic \( p ::= b \mid p \land p \mid p \lor p \mid p \to p \)
- Define a proof system (a way to prove propositions)

But it turns out we did that too!

Slogans:

- “Propositions are Types”
- “Proofs are Programs”
A slight variant

Let's take the explicitly typed ST\(\lambda\)C with base types \(b_1, b_2, \ldots\), 
no constants, pairs, and sums

\[
e \ ::= \ x \mid \lambda x.\ e \mid e \ e \\
    \mid (e, e) \mid e.1 \mid e.2 \\
    \mid A(e) \mid B(e) \mid \text{match } e \text{ with } A x.\ e \mid B x.\ e
\]

\[
\tau \ ::= \ b \mid \tau \to \tau \mid \tau \ast \tau \mid \tau + \tau
\]

Even without constants, plenty of terms type-check with \(\Gamma = \cdot\ldots\).
Example programs

\[ \lambda x : b_17. \ x \]

has type

\[ b_{17} \rightarrow b_{17} \]
Example programs

\[ \lambda x: b_1. \lambda f: b_1 \rightarrow b_2. f \; x \]

has type

\[ b_1 \rightarrow (b_1 \rightarrow b_2) \rightarrow b_2 \]
Example programs

\[
\lambda x: b_1 \to b_2 \to b_3. \lambda y : b_2. \lambda z: b_1. x \; z \; y
\]

has type

\[
(b_1 \to b_2 \to b_3) \to b_2 \to b_1 \to b_3
\]
Example programs

\[ \lambda x : b_1. \ (A(x), A(x)) \]

has type

\[ b_1 \rightarrow ((b_1 + b_7) \ast (b_1 + b_4)) \]
Example programs

$$\lambda f : b_1 \to b_3. \lambda g : b_2 \to b_3. \lambda z : b_1 + b_2. (\text{match } z \text{ with } A x. f x \mid B x. g x)$$

has type

$$(b_1 \to b_3) \to (b_2 \to b_3) \to (b_1 + b_2) \to b_3$$
Example programs

\[ \lambda x : b_1 \ast b_2. \lambda y : b_3. ((y, x.1), x.2) \]

has type

\[ (b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2) \]
Empty and Nonempty Types

So we have seen several “nonempty” types (closed terms of that type):

\[ b_{17} \rightarrow b_{17} \]
\[ b_1 \rightarrow (b_1 \rightarrow b_2) \rightarrow b_2 \]
\[ (b_1 \rightarrow b_2 \rightarrow b_3) \rightarrow b_2 \rightarrow b_1 \rightarrow b_3 \]
\[ b_1 \rightarrow (((b_1 + b_7) \ast (b_1 + b_4))) \]
\[ (b_1 \rightarrow b_3) \rightarrow (b_2 \rightarrow b_3) \rightarrow (b_1 + b_2) \rightarrow b_3 \]
\[ (b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2) \]

But there are also lots of “empty” types (no closed term of that type):

\[ b_1 \quad b_1 \rightarrow b_2 \quad b_1 + (b_1 \rightarrow b_2) \quad b_1 \rightarrow (b_2 \rightarrow b_1) \rightarrow b_2 \]

And “I” have a “secret” way of knowing whether a type will be empty; let me show you propositional logic...
# Propositional Logic

With $\rightarrow$ for implies, $\lor$ for inclusive-or and $\land$ for and:

\[
\begin{align*}
p & ::= \, b \mid p \rightarrow p \mid p \land p \mid p \lor p \\
\Gamma & ::= \, \cdot \mid \Gamma, p
\end{align*}
\]

$$\Gamma \vdash p$$

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 & \quad \Gamma \vdash p_2 \\
\end{align*}
}{
\Gamma \vdash p_1 \land p_2}
\]

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 \land p_2 & \\
\end{align*}
}{
\Gamma \vdash p_1}
\]

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 \land p_2 & \\
\end{align*}
}{
\Gamma \vdash p_2}
\]

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 & \\
\end{align*}
}{
\Gamma \vdash p_1 \lor p_2}
\]

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 & \\
\Gamma \vdash p_2 \\
\end{align*}
}{
\Gamma \vdash p_1 \lor p_2}
\]

\[
\frac{
\begin{align*}
\Gamma, p_1 \vdash p_2 \\
\end{align*}
}{
\Gamma \vdash p_1 \rightarrow p_2}
\]

\[
\frac{
\begin{align*}
\Gamma \vdash p_1 \rightarrow p_2 & \\
\Gamma \vdash p_1 \\
\end{align*}
}{
\Gamma \vdash p_2}
\]
Guess what!!!!

That’s exactly our type system, erasing terms and changing every $\tau$ to a $p$

$\Gamma \vdash e : \tau$

$\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2$

$\Gamma \vdash (e_1, e_2) : \tau_1 \ast \tau_2$

$\Gamma \vdash e : \tau_1 \ast \tau_2$

$\Gamma \vdash e : \tau_1 \ast \tau_2$

$\Gamma \vdash e : \tau_1$

$\Gamma \vdash e_1 : \tau_1$

$\Gamma \vdash e_2 : \tau_2$

$\Gamma \vdash A(e) : \tau_1 + \tau_2$

$\Gamma \vdash e : \tau_1 + \tau_2$

$\Gamma \vdash B(e) : \tau_1 + \tau_2$

$\Gamma, x : \tau_1 \vdash e_1 : \tau$

$\Gamma, y : \tau_2 \vdash e_2 : \tau$

$\Gamma \vdash \text{match } e \text{ with } A x. e_1 \mid B y. e_2 : \tau$

$\Gamma(x) = \tau$

$\Gamma, x : \tau_1 \vdash e : \tau_2$

$\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1$

$\Gamma \vdash e_2 : \tau_2$

$\Gamma \vdash e_1 e_2 : \tau_1$
Curry-Howard Isomorphism

- Given a closed term that type-checks, we can take the typing derivation, erase the terms, and have a propositional-logic proof.
- Given a propositional-logic proof, there exists a closed term with that type.
- A term that type-checks is a proof — it tells you exactly how to derive the logic formula corresponding to its type.
- Intuitionistic (hold that thought) propositional logic and simply-typed lambda-calculus with pairs and sums are the same thing.
  - Computation and logic are deeply connected
  - $\lambda$ is no more or less made up than implication
- Let’s revisit our examples under the logical interpretation...
Example proofs

\[ \lambda x : b_{17}. \ x \]

is a proof that

\[ b_{17} \rightarrow b_{17} \]
Example proofs

\[ \lambda x : b_1. \lambda f : b_1 \rightarrow b_2. f \ x \]

is a proof that

\[ b_1 \rightarrow (b_1 \rightarrow b_2) \rightarrow b_2 \]
Example proofs

\[ \lambda x : b_1 \to b_2 \to b_3 \cdot \lambda y : b_2 \cdot \lambda z : b_1 \cdot x \ z \ y \]

is a proof that

\[ (b_1 \to b_2 \to b_3) \to b_2 \to b_1 \to b_3 \]
Example proofs

\[ \lambda x : b_1. \ (A(x), A(x)) \]

is a proof that

\[ b_1 \rightarrow ((b_1 + b_7) \ast (b_1 + b_4)) \]
Example proofs

\[ \lambda f : b_1 \to b_3. \lambda g : b_2 \to b_3. \lambda z : b_1 + b_2. \]

\((\text{match } z \text{ with } A x. f x | B x. g x)\)

is a proof that

\[(b_1 \to b_3) \to (b_2 \to b_3) \to (b_1 + b_2) \to b_3\]
Example proofs

\[ \lambda x : b_1 \ast b_2. \lambda y : b_3. ((y, x.1), x.2) \]

is a proof that

\[ (b_1 \ast b_2) \rightarrow b_3 \rightarrow ((b_3 \ast b_1) \ast b_2) \]
Why care?

Because:

• This is just fascinating (glad I’m not a dog).
• For decades these were separate fields.
• Thinking “the other way” can help you know what’s possible/impossible
• Can form the basis for automated theorem provers
• Type systems should not be *ad hoc* piles of rules!

So, every typed $\lambda$-calculus is a proof system for some logic...

Is ST$\lambda$C with pairs and sums a *complete* proof system for propositional logic? Almost...
Classical vs. Constructive

Classical propositional logic has the "law of the excluded middle":

\[ \Gamma \vdash p_1 + (p_1 \rightarrow p_2) \]

(Think "\( p \) or not \( p \)" – also equivalent to double-negation.)

ST\( \lambda \)C has no proof for this; there is no expression with this type.

Logics without this rule are called constructive. They’re useful because proofs “know how the world is” and “are executable” and “produce examples”.

You can still “branch on possibilities”:

\[ ((p_1 + (p_1 \rightarrow p_2)) \ast (p_1 \rightarrow p_3) \ast ((p_1 \rightarrow p_2) \rightarrow p_3)) \rightarrow p_3 \]
Example classical proof

Theorem: I can always wake up at 9AM and get to campus by 10AM.

Proof: If it is a weekday, I can take a bus that leaves at 9:30AM. If it is not a weekday, traffic is light and I can drive. Since it is a weekday or not a weekday, I can get to campus by 10AM.

Problem: If you wake up and don’t know if it’s a weekday, this proof does not let you construct a plan to get to campus by 10AM.

In constructive logic, that never happens. You can always extract a program from a proof that “does” what you proved “could be”.

You could not prove the theorem above, but you could prove, “If I know whether it is a weekday or not, then I can get to campus by 10AM.”
Fix

A “non-terminating proof” is no proof at all.

Remember the typing rule for fix:

$$
\Gamma \vdash e : \tau \rightarrow \tau \\
\hline
\Gamma \vdash \text{fix } e : \tau
$$

That let’s us prove anything! For example: \text{fix } \lambda x : b_3. x has type \(b_3\).

So the “logic” is \text{inconsistent} (and therefore worthless)

Related: In ML, a value of type ’a never terminates normally (raises an exception, infinite loop, etc.)

```
let rec f x = f x
let z = f 0
```
Last word on Curry-Howard

It’s not just STλC and intuitionistic propositional logic.

Every logic has a corresponding typed λ calculus (and no consistent logic has something like fix).

- Example: When we add universal types (“generics”) in a few lectures, that corresponds to adding universal quantification.

If you remember one thing: the typing rule for function application is *modus ponens*. 
## Toward Evaluation Contexts

(untyped) \( \lambda \)-calculus with extensions has lots of “boring inductive rules”:

\[
\begin{align*}
  e_1 & \to e'_1 \\
  e_1 e_2 & \to e'_1 e_2 \\
  e & \to e' \\
  e & \to e'
\end{align*}
\]

\[
\begin{align*}
  v e_2 & \to v e'_2 \\
  e.1 & \to e'.1 \\
  e.2 & \to e'.2
\end{align*}
\]

\[
\begin{align*}
  (e_1, e_2) & \to (e'_1, e_2) \\
  (v_1, e_2) & \to (v_1, e'_2) \\
  A(e) & \to A(e') \\
  B(e) & \to B(e')
\end{align*}
\]

\[
\begin{align*}
  e & \to e'
\end{align*}
\]

**match** \( e \) with \( Ax. e_1 | By. e_2 \) \( \to \) **match** \( e' \) with \( Ax. e_1 | By. e_2 \)

and some “interesting do-work rules”: 

\[
\begin{align*}
  (\lambda x. e) v & \to e[v/x] \\
  (v_1, v_2).1 & \to v_1 \\
  (v_1, v_2).2 & \to v_2
\end{align*}
\]

**match** \( A(v) \) with \( Ax. e_1 | By. e_2 \) \( \to \) \( e_1[v/x] \)

**match** \( B(v) \) with \( Ay. e_1 | Bx. e_2 \) \( \to \) \( e_2[v/x] \)
Evaluation Contexts

We can define *evaluation contexts*, which are expressions with one hole where “interesting work” may occur:

\[
E ::= \cdot | E\ e | v\ E | (E, e) | (v, E) | E.1 | E.2 \\
| A(E) | B(E) | (\text{match } E \text{ with } A\ x.\ e_1 | B\ y.\ e_2)
\]

Define “filling the hole” \( E[e] \) in the obvious way (see ML code).

Semantics is now just “interesting work” rules (written \( e \xrightarrow{p} e' \)) and:

\[
\begin{align*}
  e & \xrightarrow{p} e' \\
  \hline
  E[e] & \rightarrow E[e']
\end{align*}
\]

So far, just concise notation pushing the work to *decomposition*: Given \( e \), find an \( E, e_a, e'_a \) such that \( e = E[e_a] \) and \( e_a \xrightarrow{p} e'_a \).

Theorem (Unique Decomposition): If \( \cdot \vdash e : \tau \), then \( e \) is a value or there is exactly one decomposition of \( e \).
Second Implementation

So far two interpreters:

- Old-fashioned small-step: derive a step, and iterate
- Evaluation-context small-step: decompose, fill the whole with the result of the primitive-step, and iterate

Decomposing “all over” each time is awfully redundant (as is the old-fashioned build a full-derivation of each step).

Will be more efficient in next lecture...