Asymmetric Cryptography

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Goals for Today

- Asymmetric Cryptography
- Lab 3 this week
- HW 2 also announced this week.
- There will also be an extra credit HW assignment (can only help your grade)
Requirements for Public-Key Encryption

- **Key generation:** computationally easy to generate a pair (public key PK, private key SK)
  - Computationally infeasible to determine private key SK given only public key PK

- **Encryption:** given plaintext M and public key PK, easy to compute ciphertext $C = E_{PK}(M)$

- **Decryption:** given ciphertext $C = E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to compute M from C without SK
  - Even infeasible to learn partial information about M
  - **Trapdoor function:** Decrypt(SK,Encrypt(PK,M)) = M
Some Number Theory Facts

- Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the $[1,n]$ interval that are relatively prime to $n$
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1

- Euler’s theorem:
  if $a \in \mathbb{Z}_n^*$, then $a^{\varphi(n)} \equiv 1 \text{ mod } n$

$\mathbb{Z}_n^*$: multiplicative group of integers mod $n$ (integers relatively prime to $n$)

- Special case: Fermat’s Little Theorem
  if $p$ is prime and $\gcd(a,p) = 1$, then $a^{p-1} \equiv 1 \text{ mod } p$
RSA Cryptosystem

[Rivest, Shamir, Adleman 1977]

Key generation:
- Generate large primes p, q
  - Say, 1024 bits each (need primality testing, too)
- Compute n=pq and \( \varphi(n) = (p-1)(q-1) \)
- Choose small e, relatively prime to \( \varphi(n) \)
  - Typically, e=3 or e=2^{16}+1=65537 (why?)
- Compute unique d such that ed = 1 mod \( \varphi(n) \)
- Public key = (e,n); private key = (d,n)

Encryption of m: \( c = m^e \mod n \)
- Modular exponentiation by repeated squaring

Decryption of c: \( c^d \mod n = (m^e)^d \mod n = m \)
Why RSA Decryption Works

- \( e \cdot d = 1 \mod \varphi(n) \), thus \( e \cdot d = 1 + k \cdot \varphi(n) \) for some \( k \)
  
  Can rewrite: \( e \cdot d = 1 + k(p-1)(q-1) \)

- Let \( m \) be any integer in \( \mathbb{Z}_n \)
- If \( \gcd(m, p) = 1 \), then \( m^{ed} = m \mod p \)
  - By Fermat’s Little Theorem, \( m^{p-1} = 1 \mod p \)
  - Raise both sides to the power \( k(q-1) \) and multiply by \( m \)
  - \( m^{1+k(p-1)(q-1)} = m \mod p \), thus \( m^{ed} = m \mod p \)
  - By the same argument, \( m^{ed} = m \mod q \)

- Since \( p \) and \( q \) are distinct primes and \( p \cdot q = n \),
  
  \( m^{ed} = m \mod n \) (using the Chinese Remainder Theorem)

- True for all \( m \) in \( \mathbb{Z}_n \), not just \( m \) in \( \mathbb{Z}_n^* \)
Why Is RSA Secure?

- **RSA problem**: given \( n = pq \), \( e \) such that \( \gcd(e, (p-1)(q-1)) = 1 \) and \( c \), find \( m \) such that \( m^e = c \mod n \)
  - i.e., recover \( m \) from ciphertext \( c \) and public key \((n,e)\) by taking \( e^{th} \) root of \( c \)
  - There is no known efficient algorithm for doing this

- **Factoring** problem: given positive integer \( n \), find primes \( p_1, \ldots, p_k \) such that \( n = p_1^{e_1}p_2^{e_2} \ldots p_k^{e_k} \)

- If factoring is easy, then RSA problem is easy (because knowing factors means you can compute \( d \)), but there is no known reduction from factoring to RSA
  - It may be possible to break RSA without factoring \( n \) -- but if it is, we don’t know how
On RSA encryption

- Encrypted message needs to be in interpreted as an integer less than $n$
  - Reason: Otherwise can’t decrypt.
  - Message is very often a symmetric encryption key.
- But still not quite that simple
Caveats

◆ $e = 3$ is a common exponent
  - If $m < n^{1/3}$, then $c = m^3 < n$ and can just take the cube root of $c$ to recover $m$ (i.e., no operations taken modulo $n$)
    - Even problems if “pad” $m$ in some ways [Hastad]
  - Let $c_i = m^3 \mod n_i$ - same message is encrypted to three people
    - Adversary can compute $m^3 \mod n_1n_2n_3$ (using CRT)
    - Then take ordinary cube root to recover $m$

◆ Don’t use RSA directly for privacy! Need to preprocess input in some way.
Sample Encryption

- 26 2 15 13 7 14 13 13 1 28 14 15 13 14 20 9 6 31 25 26 14 16 23 15 26 2 6 13 1

- P=3, Q=11, N=33, E=7, D=3
- ‘A’ converted to 1 before encryption; ‘B’ Converted to 2 before encryption; ...


- http://www.wolframalpha.com/
Integrity in RSA Encryption

- Plain RSA does not provide integrity
  - Given encryptions of $m_1$ and $m_2$, attacker can create encryption of $m_1 \cdot m_2$
    - $(m_1^e) \cdot (m_2^e) \mod n = (m_1 \cdot m_2)^e \mod n$
  - Attacker can convert $m$ into $m^k$ without decrypting
    - $(m_1^e)^k \mod n = (m^k)^e \mod n$

- In practice, OAEP is used: instead of encrypting $M$, encrypt $M \oplus G(r) \oplus H(M \oplus G(r))$
  - $r$ is random and fresh, $G$ and $H$ are hash functions
  - Resulting encryption is plaintext-aware: infeasible to compute a valid encryption without knowing plaintext
    - ... if hash functions are “good” and RSA problem is hard
OAEP (image from PKCS #1 v2.1)

\[ r \oplus H(M \oplus G(r)) \]

\[ M \oplus G(r) \]
Summary of RSA

- Defined RSA primitives
  - Encryption and Decryption
  - Underlying number theory
  - Practical concerns, some mis-uses
  - OAEP
Digital Signatures: Basic Idea

**Given:** Everybody knows Bob’s public key
Only Bob knows the corresponding private key

**Goal:** Bob sends a “digitally signed” message
1. To compute a signature, must know the private key
2. To verify a signature, enough to know the public key
RSA Signatures

- Public key is \((n,e)\), private key is \(d\)
- To **sign** message \(m\): \(s = m^d \mod n\)
  - Signing and decryption are the same **underlying** operation in RSA
  - It’s infeasible to compute \(s\) on \(m\) if you don’t know \(d\)

- To **verify** signature \(s\) on message \(m\):
  \[ s^e \mod n = (m^d)^e \mod n = m \]
  - Just like encryption
  - Anyone who knows \(n\) and \(e\) (public key) can verify signatures produced with \(d\) (private key)

- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures
Encryption and Signatures

- Often people think: Encryption and decryption are inverses.
- That’s a common view
  - True for the RSA \textit{primitive (underlying component)}
- But not one we’ll take
  - To really use RSA, we need padding
  - And there are many other decryption methods
  - And there are many other signing methods
Digital Signature Standard (DSS)

◆ U.S. government standard (1991-94)
  • Modification of the ElGamal signature scheme (1985)

◆ Key generation:
  • Generate large primes $p, q$ such that $q$ divides $p-1$
    $-2^{159} < q < 2^{160}, 2^{511+64t} < p < 2^{512+64t}$ where $0 \leq t \leq 8$
  • Select $h \in \mathbb{Z}_p^*$ and compute $g = h^{(p-1)/q} \mod p$
  • Select random $x$ such $1 \leq x \leq q-1$, compute $y = g^x \mod p$

◆ Public key: $(p, q, g, y = g^x \mod p)$, private key: $x$

◆ Security of DSS requires hardness of discrete log
  • If could solve discrete logarithm problem, would extract $x$ (private key) from $g^x \mod p$ (public key)
DSS: Signing a Message (Skim)

- **Message Hash function (SHA-1)**
- **Random secret** between 0 and q
- **Compute** \( r = (g^k \mod p) \mod q \)
- **Private key**
- **Hash function**
- **Compute** \( s = k^{-1} \cdot (H(M) + x \cdot r) \mod q \)

\((r,s)\) is the signature on M
DSS: Verifying a Signature (Skim)

Compute \( w = s'^{-1} \mod q \)

Compute \( (g^{H(M')} \cdot y^{r'w} \mod q \mod p) \mod q \)

If they match, signature is valid
Advantages of Public-Key Crypto

❖ Confidentiality without shared secrets
  - Very useful in open environments
  - No “chicken-and-egg” key establishment problem
    – With symmetric crypto, two parties must share a secret before they can exchange secret messages
    – Caveats to come

❖ Authentication without shared secrets
  - Use digital signatures to prove the origin of messages

❖ Reduce protection of information to protection of authenticity of public keys
  - No need to keep public keys secret, but must be sure that Alice’s public key is really her true public key
Disadvantages of Public-Key Crypto

- Calculations are 2-3 orders of magnitude slower
  - Modular exponentiation is an expensive computation
  - Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
    - E.g., IPsec, SSL, SSH, ...
- Keys are longer
  - 1024+ bits (RSA) rather than 128 bits (AES)
- Relies on unproven number-theoretic assumptions
  - What if factoring is easy?
    - Factoring is believed to be neither P, nor NP-complete
  - (Of course, symmetric crypto also rests on unproven assumptions)
Note: Optimizing Exponentiation

How to compute $M^x \mod N$? Say $x=13$

Sums of power of 2, $x = 8+4+1 = 2^3+2^2+2^0$

Can also write $x$ in binary, e.g., $x = 1101$

Can solve by repeated squaring

- $y = 1$
- $y = y^2 \cdot M \mod N$ // $y = M$
- $y = y^2 \cdot M \mod N$ // $y = M^2 \cdot M = M^{2+1} = M^3$
- $y = y^2 \mod N$ // $y = (M^{2+1})^2 = M^{4+2}$
- $y = y^2 \cdot M \mod N$ // $y = (M^{4+2})^2 \cdot M = M^{8+4+1}$

Does anyone see a potential issue?
Timing attacks

Collect timings for exponentiation with a bunch of messages $M_1$, $M_2$, ... (e.g., RSA signing operations with a private exponent)

Assume (inductively) know $b_3=1$, $b_2=1$, guess $b_1=1$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$b_i = 0$</th>
<th>$b_i = 1$</th>
<th>Comp</th>
<th>Meas</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>$y = y^2 \mod N$</td>
<td>$y = y^2 \cdot M_1 \mod N$</td>
<td></td>
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<tr>
<td>2</td>
<td>$y = y^2 \mod N$</td>
<td>$y = y^2 \cdot M_1 \mod N$</td>
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<tr>
<td>1</td>
<td>$y = y^2 \mod N$</td>
<td>$y = y^2 \cdot M_1 \mod N$</td>
<td>$X_1$ secs</td>
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<tr>
<td>0</td>
<td>$y = y^2 \mod N$</td>
<td>$y = y^2 \cdot M_1 \mod N$</td>
<td></td>
<td>$Y_1$ secs</td>
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<td>$y = y^2 \cdot M_2 \mod N$</td>
<td>$X_2$ secs</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$y = y^2 \mod N$</td>
<td>$y = y^2 \cdot M_2 \mod N$</td>
<td></td>
<td>$Y_2$ secs</td>
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</table>
Timing attacks

- If $b_1 = 1$, then set of $\{ Y_j - X_j \mid j \in \{1,2, \ldots\} \}$ has distribution with “small” variance (due to time for final step, $i=0$)
  - “Guess” was correct when we computed $X_1, X_2, \ldots$
- If $b_1 = 0$, then set of $\{ Y_j - X_j \mid j \in \{1,2, \ldots\} \}$ has distribution with “large” variance (due to time for final step, $i=0$, and incorrect guess for $b_1$)
  - “Guess” was incorrect when we computed $X_1, X_2, \ldots$
  - So time computation wrong ($X_j$ computed as large, but really small, ...)
- Strategy: Force user to sign large number of messages $M_1, M_2, \ldots$. Record timings for signing.
- Iteratively learn bits of key by using above property.