CSE 473: Artificial Intelligence

Bayesian Networks

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Many slides over the course adapted from either Luke Zettlemoyer, Pieter Abbeel, Dan Klein, Stuart Russell or Andrew Moore

Outline

- Probabilistic models (and inference)
  - Bayesian Networks (BNs)
  - Independence in BNs

Probabilistic Models

- Models describe how (a portion of) the world works
- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box
- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information

Independence

- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

  - This says that their joint distribution *factors* into a product two simpler distributions
  - Another form:

$$\forall x, y : P(x|y) = P(x)$$

  - We write: $X \perp\!\!\!\perp Y$

- Independence is a simplifying *modeling assumption*
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity, Toothache}?

Example: Independence

- N fair, independent coin flips:

$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

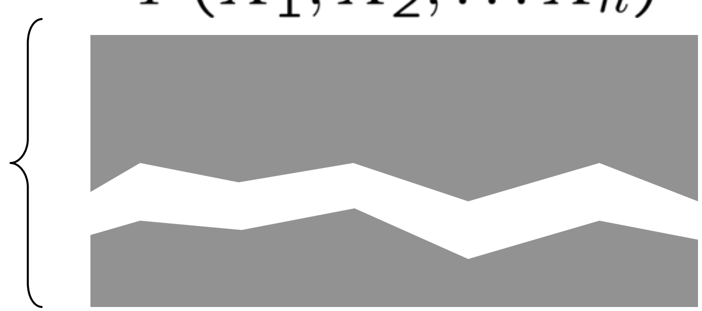
$\cdots$

$P(X_n)$

h	0.5
t	0.5

$P(X_1, X_2, \ldots X_n)$

2^n



Conditional Independence

- Unconditional (absolute) independence very rare (why?)

- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$
$$\forall x, y, z : P(x|z, y) = P(x|z)$$

$$X \perp\!\!\!\perp Y | Z$$

  - What about these domain:
    - Traffic, Umbrella, Raining
    - Toothache, Cavity, Catch

Conditional Independence and the Chain Rule

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes' nets/ graphical models help us express conditional independence assumptions

Ghostbusters Chain Rule

- 2-position maze, each sensor indicates ghost location
- T: Top square is red
  B: Bottom square is red
  G: Ghost is in the top
- That means, the two sensors are conditionally independent, given the ghost position
- Can assume:
  $P(+g) = 0.5$
  $P(+t \mid +g) = 0.8$
  $P(+t \mid \neg g) = 0.4$
  $P(+b \mid +g) = 0.4$
  $P(+b \mid \neg g) = 0.8$

$$P(T,B,G) = P(G)\, P(T|G)\, P(B|G)$$

T	B	G	P(T,B,G)
+t	+b	+g	0.16
+t	+b	¬g	0.16
+t	¬b	+g	0.24
+t	¬b	¬g	0.04
¬t	+b	+g	0.04
¬t	+b	¬g	0.24
¬t	¬b	+g	0.06
¬t	¬b	¬g	0.06

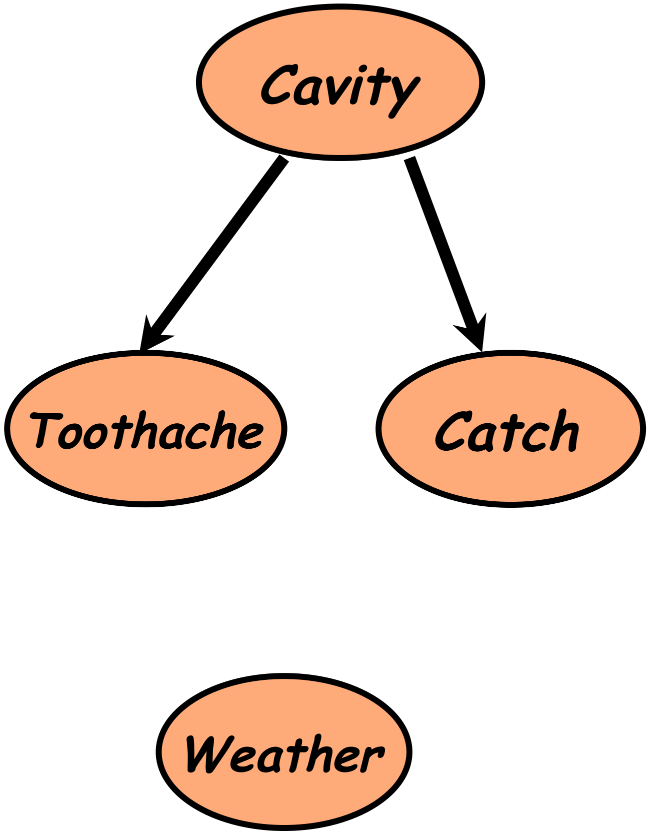
Bayes' Nets: Big Picture

- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time

- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions

Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or
  - unassigned (unobserved)
- Arcs: interactions
  - Indicate "direct influence" between variables
  - Formally: encode conditional independence (more later)

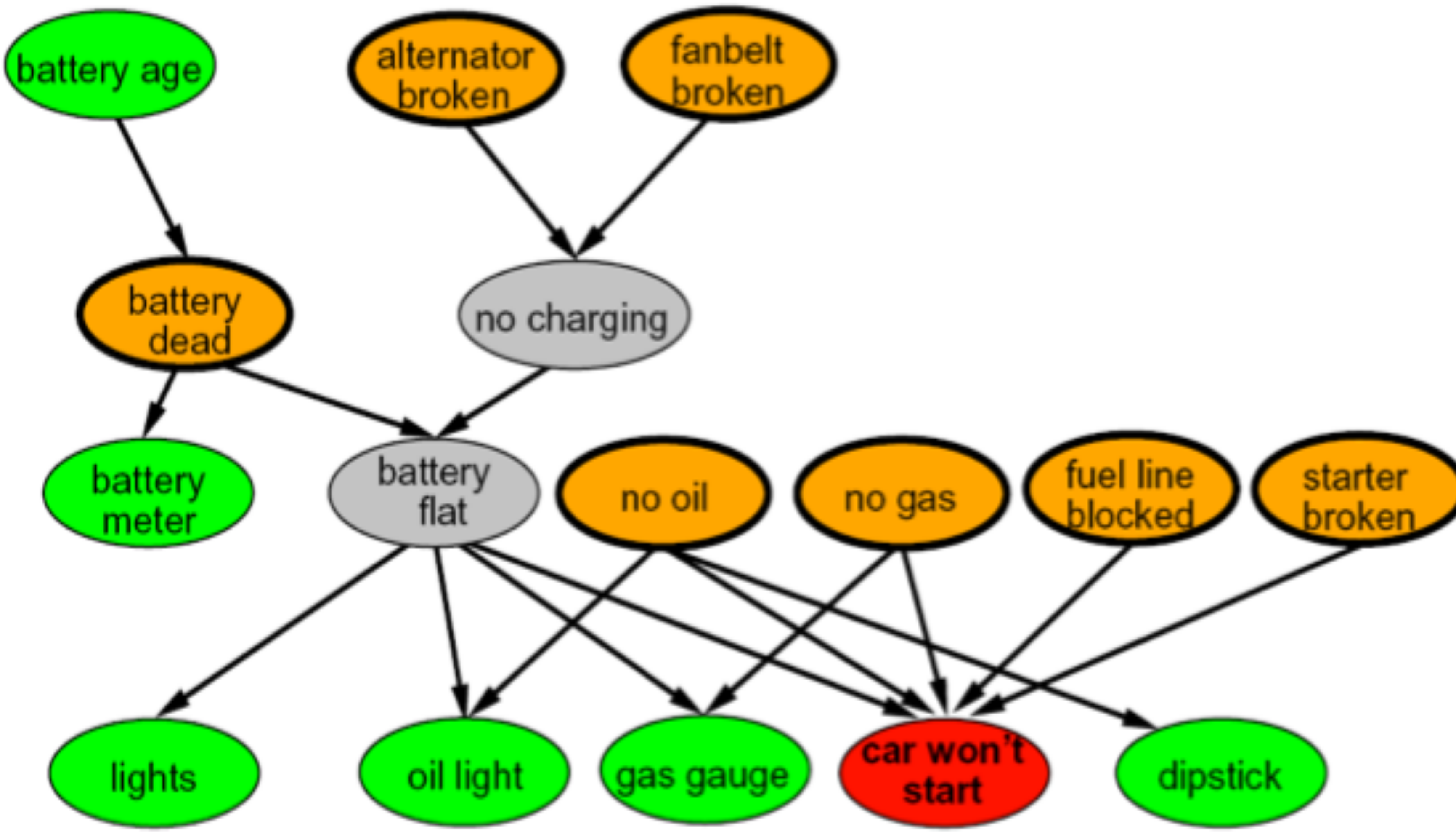


Example: Flip Coins

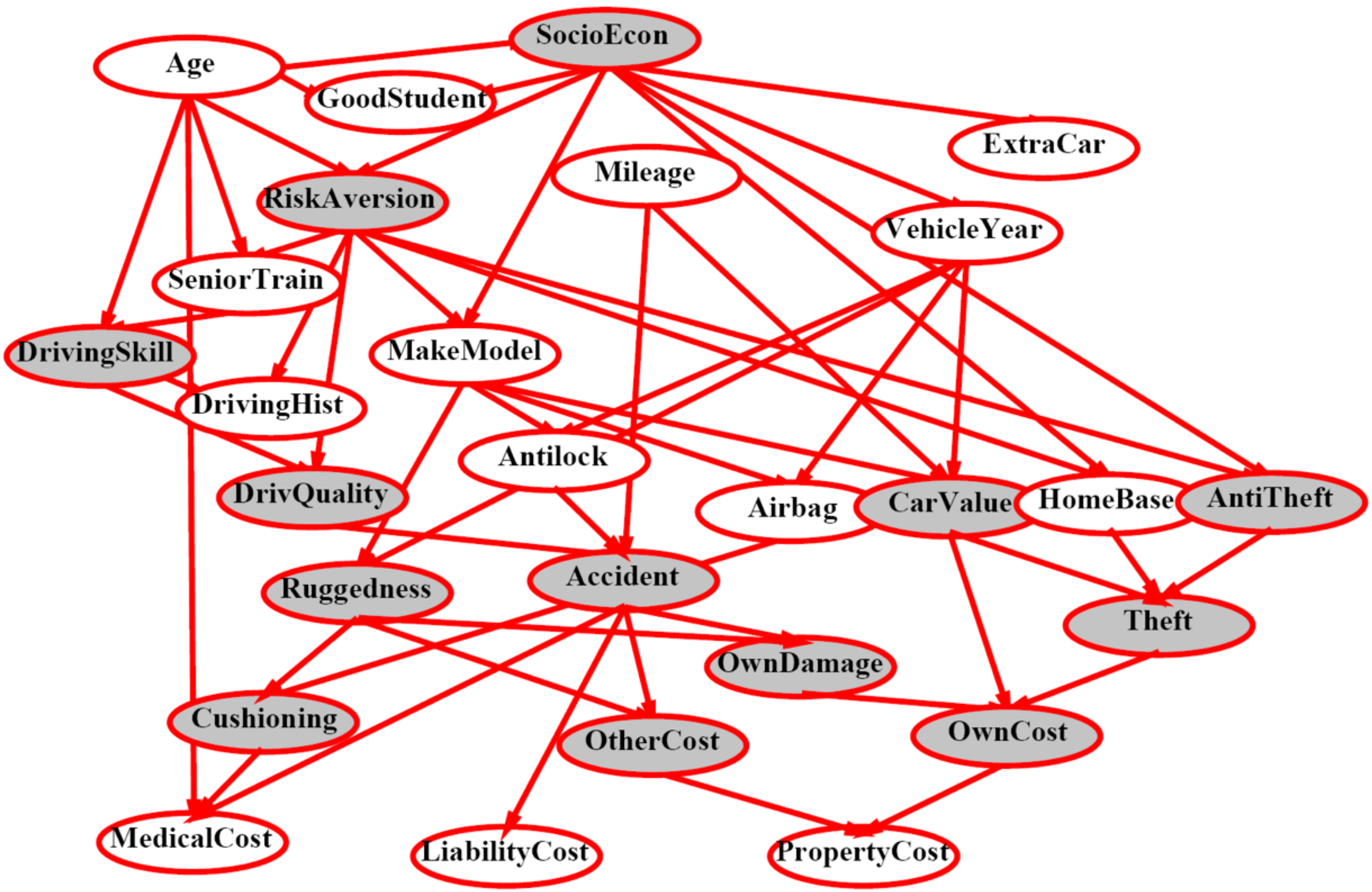
- N independent flip coins

- No interactions between variables
  - Absolute independence

Example Bayes' Net: Car



Example Bayes' Net: Insurance



Example: Traffic

- Variables:
  - R: It rains
  - T: There is traffic
- Model 1: independence
- Model 2: rain is conditioned on traffic
  - Why is an agent using model 2 better?
- Model 3: traffic is conditioned on rain
  - Is this better than model 2?

Example: Traffic II

- Let's build a graphical model

- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity

Example: Alarm Network

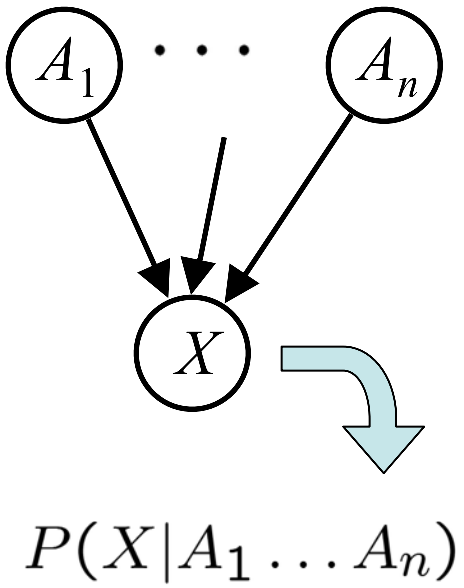
- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!

Bayes' Net Semantics

- Let's formalize the semantics of a Bayes' net
- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

  - CPT: conditional probability table



A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

- This lets us reconstruct any entry of the full joint
- Not every BN can represent every joint distribution
  - The topology enforces certain *independence* assumptions
  - Compare to the exact decomposition according to the chain rule!

Probabilities in BN

- Why are we guaranteed that setting

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$
- Assume conditional independences: $P(x_i | x_1, \ldots x_{i-1}) = P(x_i | parents(X_i))$

→ Consequence: $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$

- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

Bayes Net Probabilities

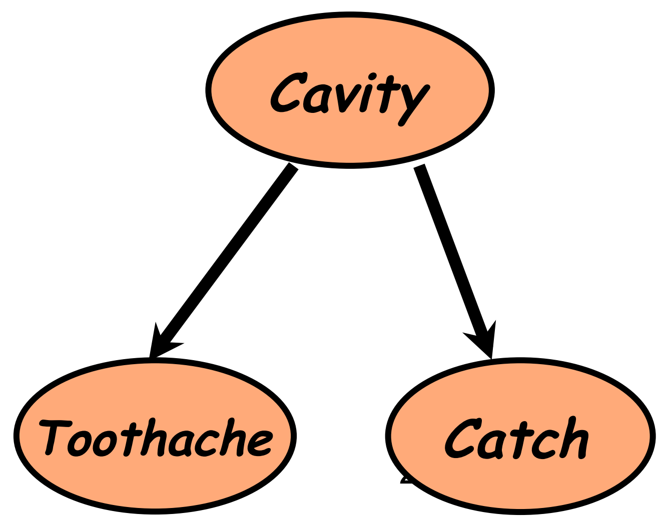
- Bayes nets compactly represent joint distributions (instead of big joint table)
  - A joint distribution using chain rule

$$P(x_1 \ldots x_n) = \prod_i P(x_i \mid parents(x_i))$$

- {Cavity, Toothache, Catch}

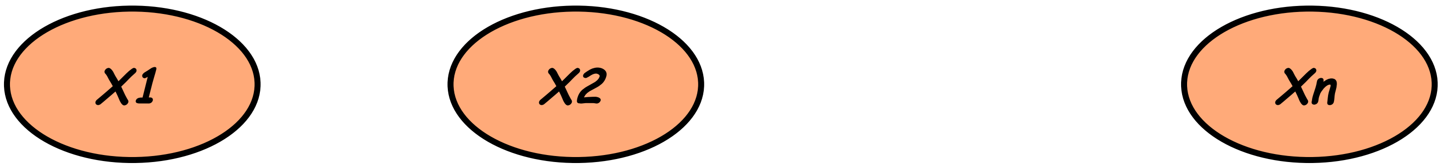
  P(Cavity, Toothache, ~Catch) ?

P(Cavity, Toothache, ~Catch) =
P(cavity)P(toothache|cavity)
P(~catch|cavity)



Example: Flip Coins

- N independent flip coins



	P
Head	0.5
Tail	0.5

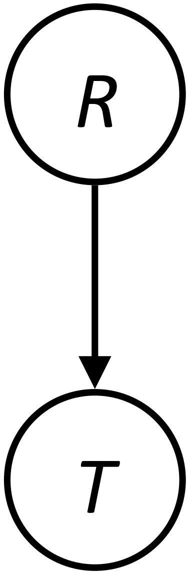
	P
Head	0.5
Tail	0.5

	P
Head	0.5
Tail	0.5

- P(h,h,t,h)?

  - No interactions between variables: absolute independence

Example: Traffic



$P(R)$

+r	1/4
-r	3/4

$P(+r, -t) =$

$P(T|R)$

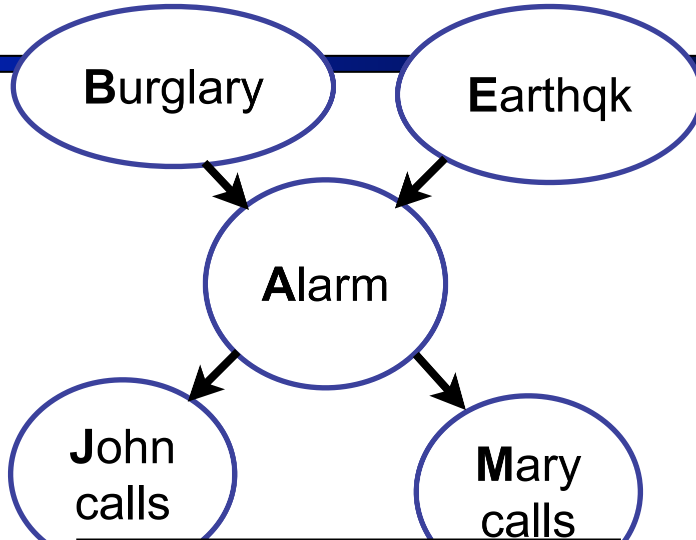
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Example: Alarm Network

B	P(B)
+b	0.001
¬b	0.999



E	P(E)
+e	0.002
¬e	0.998

A	J	P(J|A)
+a	+j	0.9
+a	¬j	0.1
¬a	+j	0.05
¬a	¬j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	¬m	0.3
¬a	+m	0.01
¬a	¬m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	¬a	0.05
+b	¬e	+a	0.94
+b	¬e	¬a	0.06
¬b	+e	+a	0.29
¬b	+e	¬a	0.71
¬b	¬e	+a	0.001
¬b	¬e	¬a	0.999

$$P(+b, -e, +a, -j, +m) =$$
$$P(+b)P(-e)P(+a| + b, -e)P(-j| + a)P(+m| + a) =$$
$$0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7$$

Changing Bayes' Net Structure

- The same joint distribution can be encoded in many different Bayes' nets

- Analysis question: given some edges, what other edges do you need to add?
  - One answer: fully connect the graph
  - Better answer: don't make any false conditional independence assumptions

Example: Independence

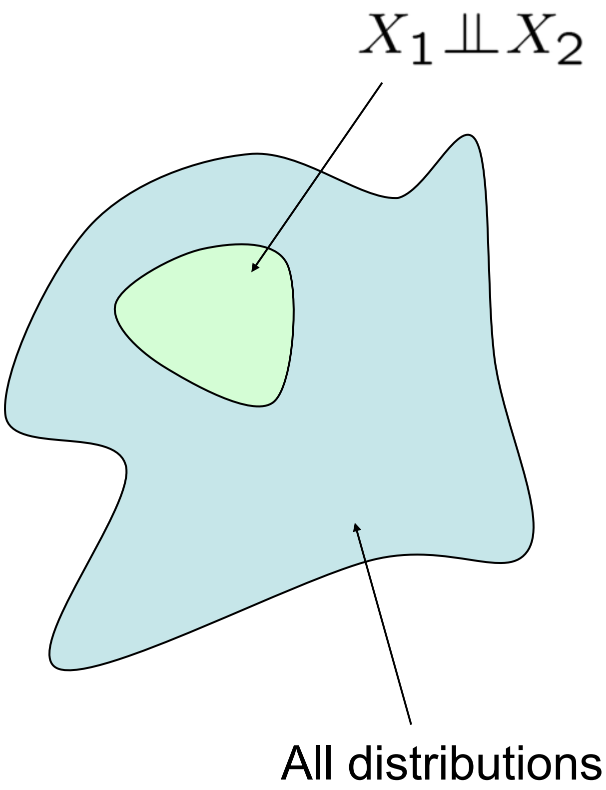
- For this graph, you can fiddle with θ (the CPTs) all you want, but you won't be able to represent any distribution in which the flips are dependent!

$P(X_1)$

h	0.5
t	0.5

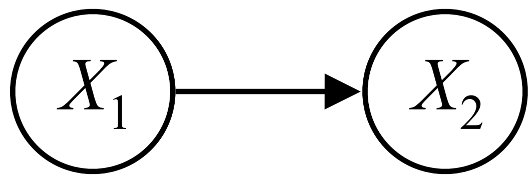
$P(X_2)$

h	0.5
t	0.5



Example: Coins

- Extra arcs don't prevent representing independence, just allow non-independence



$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$P(X_1)$

h	0.5
t	0.5

$P(X_2|X_1)$

h | h	0.5
t | h	0.5
h | t	0.5
t | t	0.5

- Adding unneeded arcs isn't wrong, it's just inefficient

Size of a Bayes Net

- How big is a joint distribution over N Boolean variables?

  2^N

- How big is an N-node net if nodes have up to k parents?

  $O(N * 2^{k+1})$

- Both give you the power to calculate

$$P(X_1, X_2, \ldots X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)

Bayes Nets

- So far: how a Bayes' net encodes a joint distribution

- Next: how to answer queries about that distribution
  - Today:
    - First assembled BNs using an intuitive notion of conditional independence as causality
    - Then saw that key property is conditional independence
  - Main goal: answer queries about conditional independence and influence

- After that: how to answer numerical queries (inference)