What’s on our menu today?

First-Order Logic
  • Definitions
  • Universal and Existential Quantifiers
  • Skolemization
  • Unification
Propositional vs. First-Order

Propositional logic: Deals with facts and propositions (can be true or false):

- $P_{1,1}$ “there is a pit in (1,1)"
- George_Monkey “George is a monkey”
- George_Curious “George is curious”
- 473student1_Monkey
- $(\text{George\_Monkey} \land \neg 473\text{student1\_Monkey}) \lor \ldots$

First-order logic: Deals with objects and relations

Objects: George, 473Student1, Monkey2, Raj, ...
Relations: Monkey(George), Curious(George),
- Smarter(473Student1, Monkey2)
- Smarter(Monkey2, Raj)
- Stooges(Larry, Moe, Curly)
- PokesInTheEyes(Moe, Curly)
- PokesInTheEyes(473Student1, Raj)
## FOL Definitions

### Constants
Name a specific object.
- George, Monkey2, Larry, ...

### Variables
Refer to an object without naming it.
- X, Y, ...

### Relations (predicates)
Properties of or relationships between objects.
- Curious, PokesInTheEyes, ...

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### Functions
Mapping from objects to objects.
- banana-of, grade-of, binders-full-of

### Terms
Logical expressions referring to objects
- banana-of(George)
- grade-of(stdnt1)
- binders-full-of(women)
- binders-full-of(men)
- binders-full-of(monkeys)
More Definitions

Logical connectives: and, or, not, ⇒, ⇔

Quantifiers:

• ∀ For all (Universal quantifier)
• ∃ There exists (Existential quantifier)

Examples

• George is a monkey and he is curious
  \text{Monkey}(George) \land \text{Curious}(George)

• All monkeys are curious
  \forall m: \text{Monkey}(m) \Rightarrow \text{Curious}(m)

• There is a curious monkey
  \exists m: \text{Monkey}(m) \land \text{Curious}(m)

Quantifier / Connective Interaction

\text{M}(x) = \text{“x is a monkey”}
\text{C}(x) = \text{“x is curious”}

∀x: \text{M}(x) \land \text{C}(x)
  \text{“Everything is a curious monkey”}

∀x: \text{M}(x) \Rightarrow \text{C}(x)
  \text{“All monkeys are curious”}

∃x: \text{M}(x) \land \text{C}(x)
  \text{“There exists a curious monkey”}

∃x: \text{M}(x) \Rightarrow \text{C}(x)
  \text{“There exists an object that is either a curious monkey, or not a monkey at all”}
Nested Quantifiers: 
Order matters!

\[ \forall x \exists y \ P(x,y) \neq \exists y \forall x \ P(x,y) \]

Example

Every monkey has a tail
\[ \forall m \exists t \ has(m,t) \]

Every monkey shares a tail!
\[ \exists t \forall m \ has(m,t) \]

Try:

Everybody loves somebody vs. Someone is loved by everyone
\[ \forall x \exists y \ loves(x, y) \] vs. \[ \exists y \forall x \ loves(x, y) \]

Semantics

\textit{Semantics} = what the arrangement of symbols means in the world

Propositional logic

- Basic elements are propositional variables e.g., \( P_{1,1} \) (refer to facts about the world)
- Possible worlds: mappings from variables to T/F

First-order logic

- Basic elements are terms, e.g., George, banana-of(George), binders-full-of(banana-of(George)) (logical expressions that refer to objects)
- Interpretations: mappings from terms to real-world elements.
Example: A World of Kings and Legs

Syntactic elements:
Constants: Richard John
Functions: LeftLeg(p)
Relations: On(x, y) King(p)

Interpretation I
Interpretations map syntactic tokens to model elements
Constants: Richard John
Functions: LeftLeg(p)
Relations: On(x, y) King(p)
Interpretation II

Constants: Richard, John

Functions: LeftLeg(p)

Relations: On(x, y), King(p)

Two constants (and 5 objects in world)
- Richard, John (R, J, crown, RL, JL)

5^2 = 25 object mappings

One unary relation
King(x)
Infinite number of values for x → infinite mappings
Even if we restricted x to: R, J, crown, RL, JL:
2^5 = 32 unary truth mappings

Two binary relations
Leg(x, y): On(x, y)
Infinite. But even restricting x, y to five objects still yields 2^{25} mappings for each binary relation
### Satisifiability, Validity, & Entailment

- **S is valid** if it is true in all interpretations.
- **S is satisfiable** if it is true in some interpretation.
- **S is unsatisfiable** if it is false in all interpretations.

\[ S_1 \models S_2 \] (\( S_1 \) entails \( S_2 \)) if

- for all interpretations where \( S_1 \) is true, \( S_2 \) is also true.

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### Propositional Logic vs. First Order

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<td>May run forever if KB ( \not\models \alpha ).</td>
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First-Order Wumpus World

Objects
- Squares, wumpuses, agents,
- gold, pits, stinkiness, breezes

Relations
- Square topology (adjacency),
- Pits/breezes,
- Wumpus/stinkiness

Wumpus World: Squares

- Each square as an object:
  Square_{1,1}, Square_{1,2}, ..., Square_{3,4}, Square_{4,4}
- Square topology relations?
  Adjacent(Square_{1,1}, Square_{2,1})
  ...
  Adjacent(Square_{3,4}, Square_{4,4})

Better: Squares as lists:
[1, 1], [1, 2], ..., [4, 4]

Square topology relations:
\forall x, y, a, b: Adjacent([x, y], [a, b]) ⇔
\quad [a, b] \in \{[x+1, y], [x-1, y], [x, y+1], [x, y-1]\}
**Wumpus World: Pits**

- Each pit as an object:
  - Pit\(_{1,1}\), Pit\(_{1,2}\), ...,
  - Pit\(_{3,4}\), Pit\(_{4,4}\)
- Problem?
  - Not all squares have pits

List only the pits we have?
  - Pit\(_{3,1}\), Pit\(_{3,3}\), Pit\(_{4,4}\)

Problem?
  - No reason to distinguish pits (same properties)

Better: pit as unary predicate
  - Pit(x)
  - Pit([3,1]), Pit([3,3]), Pit([4,4]) will be true

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**Wumpus World: Breezes**

- Represent breezes like pits, as unary predicates:
  - Breezy(x)

“Squares next to pits are breezy”:
  - \(\forall c, d, a, b:\)
  - Pit([c, d]) \land Adjacent([c, d], [a, b]) \Rightarrow Breezy([a, b])\)
**Wumpus World: Wumpuses**

- **Wumpus as object:**
  Wumpus

- **Wumpus home as unary predicate:**
  WumpusIn(x)

**Better:** Wumpus’s home as a function:
Home(Wumpus) references the wumpus’s home square.

**FOL Reasoning: Outline**

Basics of FOL reasoning
Classes of FOL reasoning methods
  - Forward & Backward Chaining
  - Resolution
  - Compilation to SAT
**Basics: Universal Instantiation**

Universally quantified sentence:

- \( \forall x: \text{Monkey}(x) \Rightarrow \text{Curious}(x) \)

Intuitively, \( x \) can be anything:

- \( \text{Monkey}(\text{George}) \Rightarrow \text{Curious}(\text{George}) \)
- \( \text{Monkey}(473\text{Student}1) \Rightarrow \text{Curious}(473\text{Student}1) \)
- \( \text{Monkey}(\text{DadOf}(\text{George})) \Rightarrow \text{Curious}(\text{DadOf}(\text{George})) \)

Formally:

\[
\begin{align*}
\forall x & \ S \\
\text{Subst}(\{x/p\}, S) & \Rightarrow \text{Monkey}(\text{George}) \Rightarrow \text{Curious}(\text{George})
\end{align*}
\]

**Basics: Existential Instantiation**

Existentially quantified sentence:

\( \exists x: \text{Monkey}(x) \land \neg \text{Curious}(x) \)

Intuitively, \( x \) must name something. But what?

Can we conclude:

\( \text{Monkey}(\text{George}) \land \neg \text{Curious}(\text{George}) \) ???

No! \( S \) might not be true for \( \text{George} \)!

Use a **Skolem Constant** and draw the conclusion:

\( \text{Monkey}(K) \land \neg \text{Curious}(K) \)

where \( K \) is a completely new symbol (stands for the monkey for which the statement is true)

Formally:

\[
\begin{align*}
\exists x & \ S \\
\text{Subst}(\{x/K\}, S) & \Rightarrow K \text{ is called a Skolem constant}
\end{align*}
\]
**Basics: Generalized Skolemization**

What if our existential variable is nested?

∀x ∃y: Monkey(x) → HasTail(x, y)

Can we conclude:

∀x: Monkey(x) → HasTail(x, K_Tail) ???

Nested existential variables can be replaced by Skolem functions

• Args to function are all surrounding ∀ vars

∀x: Monkey(x) → HasTail(x, f(x))

“tail-of” function

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**Motivation for Unification**

What if we want to use modus ponens?

Propositional Logic:

\[ a \land b, \quad a \land b \Rightarrow c \]

\[ c \]

In First-Order Logic?

∀x: Monkey(x) → Curious(x)

Monkey(George)

???

Must “unify” x with George:

Need to substitute {x/George} in Monkey(x) → Curious(x) to infer Curious(George)
What is Unification?

Match up expressions by finding variable values that make the expressions identical.

Unify(x, y) returns most general unifier (MGU).

MGU places fewest restrictions on values of variables.

Examples:

- Unify(city(x), city(seattle)) returns {x/seattle}
- Unify(PokesInTheEyes(Moe, x), PokesInTheEyes(y, z)) returns {y/Moe, z/x}
  - {y/Moe, x/Moe, z/Moe} possible but not MGU
Unification and Substitution

Unification produces a mapping from variables to values (e.g., \{x/seattle, y/tacoma\})

Substitution: \text{Subst}(\text{mapping}, \text{sentence}) \text{ returns new sentence with variables replaced by values}

- Subst(\{x/seattle, y/tacoma\}), connected(x, y)), returns connected(seattle, tacoma)

Next Time

Reasoning with FOL
  Chaining
  Resolution
  Compilation to SAT

To Do:
  Project #2
  Read Chapters 8–9