1. Chapter 7, exercise 7.4.

(a) $\alpha$ is valid if and only if $\text{True} \models \alpha$.

**Forward direction:** If $\text{True} \models \alpha$, then $\alpha$ is valid.

By definition, $\text{True} \models \alpha$ means that $\alpha$ is true in all worlds where $\text{True}$ is True; this is all worlds. Thus $\alpha$ is true in all worlds, which is exactly the definition of validity. So $\alpha$ is in this case valid.

**Backward direction:** If $\alpha$ is valid, then $\text{True} \models \alpha$.

By definition, if $\alpha$ is valid then it is true in all worlds. In this case anything entails $\alpha$, so clearly $\text{True}$ entails $\alpha$.

(b) For any $\alpha$, $\text{False} \models \alpha$.

Recall the definition of entailment: $p \models q$ means that in all worlds in which $p$ is true, $q$ is true as well. So, $\text{False} \models \alpha$ means that in all worlds in which $\text{False}$ is true, $\alpha$ is true. But there are no worlds in which $\text{False}$ is true! Clearly if there are no worlds in a set, then that set satisfies the condition that $\alpha$ be true in all worlds in that set.

(c) $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

**Forward direction:** If $\alpha \models \beta$, then the sentence $(\alpha \Rightarrow \beta)$ is valid.

By definition, $\alpha \models \beta$ means that $\beta$ is true in all worlds in which $\alpha$ is true. Thus, in all worlds in which $\alpha$ is true $\alpha \Rightarrow \beta$ holds because both $\alpha$ and $\beta$ will be true. We must also consider worlds in which $\alpha$ is false; in these worlds, $\alpha \Rightarrow \beta$ also holds by definition of the falsehood of $\alpha$.

**Backward direction:** If the sentence $(\alpha \Rightarrow \beta)$ is valid, then $\alpha \models \beta$.

If the sentence $(\alpha \Rightarrow \beta)$ is valid, then it is true in all worlds. Thus, for every world, it must be the case that either both $\alpha$ and $\beta$ are true, or $\alpha$ is false. This is enough to tell us that in every world in which $\alpha$ is true, $\beta$ is also true, which is the definition of entailment.

2. Write axioms describing the following predicates: $\text{GrandChild}$, $\text{GreatGrandparent}$, $\text{Brother}$, $\text{Aunt}$, $\text{SisterInLaw}$, $\text{FirstCousin}$. You may assume that the predicates $\text{ParentOf}(x, y)$, $\text{ChildOf}(x, y)$, $\text{Male}(x)$, $\text{Female}(x)$, $\text{Married}(x, y)$ are already defined.

(a) $\text{GrandChild}: \text{GrandChild}(z, x) \Leftrightarrow \exists y(\text{ChildOf}(z, y) \land \text{ChildOf}(y, x))$

Here, $y$ is the person whose parent ($x$) is the grandparent of their child ($z$).

(b) $\text{GreatGrandparent}: \text{GreatGrandparent}(w, z) \Leftrightarrow \exists x, y(\text{ParentOf}(w, x) \land \text{ParentOf}(x, y) \land \text{ParentOf}(y, z))$
Here, $x$ and $y$ are the two generations between the great-grandparent and the great-grandchild.

(c) **Brother:** $\text{Brother}(x, y) \iff \exists a, b (\text{ParentOf}(a, x) \land \text{ParentOf}(b, x) \land \text{ParentOf}(a, y) \land \text{ParentOf}(b, y) \land \text{Male}(x) \land x \neq y)$

This answer assumes that brother means full brother: answers in which only one parent is shared (e.g. half-brother definitions) are also correct. We disallow $x = y$ because one cannot be the brother of oneself.

(d) **Aunt:** $\text{Aunt}(x, y) \iff (\exists a (\text{ParentOf}(a, y) \land \text{Sister}(x, a)) \lor (\exists a, b (\text{ParentOf}(a, y) \land \text{Brother}(b, a) \land \text{Married}(b, y)))$

In this case $\text{Sister}(x, y)$ is defined analogously with $\text{Brother}(x, y)$ in the previous line. An aunt is either the sister of $a$, who is $y$’s parent; or, an aunt is the wife of a parent’s ($a$’s) brother $b$.

(e) **SisterInLaw:** $\text{SisterInLaw}(x, y) \iff \exists a (\text{Female}(x) \land \text{Married}(x, a) \land (\text{Sister}(a, y) \lor \text{Brother}(a, y)))$

Here, $a$ is a sibling of $y$ whose wife, $x$, is $y$’s sister-in-law.

(f) **FirstCousin:** $\text{FirstCousin}(x, y) \iff \exists a, b, p, q (a \neq b \land x \neq y \land p \neq q \land \text{ParentOf}(a, x) \land \text{ParentOf}(p, a) \land \text{ParentOf}(q, a) \land \text{ParentOf}(b, y) \land \text{ParentOf}(p, b) \land \text{ParentOf}(q, b))$

Here, $p$ and $q$ are the pair of grandparents shared by first-cousins $x$ and $y$. $a$ and $b$ are siblings, and the parents of $x$ and $y$, respectively. We disallow $x = y$ because this would collapse to “I am my own first cousin”. We disallow $a = b$ because this would collapse to “I am the first cousin of my sibling”. Finally, we disallow $p = q$ to enforce that the first cousins share two grandparents.

3. Recall the nursery rhyme:

   Fuzzy Wuzzy was a bear,
   Fuzzy Wuzzy had no hair.
   Was he fuzzy?

In this exercise you will prove that Fuzzy Wuzzy is indeed fuzzy. Fuzzy Wuzzy’s universe is governed by the following six rules:

$\forall b \exists c \text{Bear}(b) \Rightarrow \text{Coat}(c) \land \text{Has}(b, c)$

Every bear owns a coat.

$\neg \exists c \text{Raincoat}(c) \land \text{Furcoat}(c)$

No coat is both a raincoat and a furcoat.

$\forall c \text{Coat}(c) \Rightarrow (\text{Raincoat}(c) \lor \text{Furcoat}(c))$

Every coat is either a raincoat or a furcoat, or both.
\[ \forall x, c \, \text{Has}(x, c) \land \text{Furcoat}(c) \Rightarrow \text{Fuzzy}(x) \]

Anything that owns a fur coat is fuzzy.

\[ \forall x \, \text{HasHair}(x) \Rightarrow \text{Fuzzy}(x) \]

Anything that has hair is fuzzy.

\[ \neg \exists c \, \text{Has}(\text{FuzzyWuzzy}, c) \land \text{Raincoat}(c) \]

Fuzzy Wuzzy doesn’t own a raincoat.

In addition, our knowledge base contains the following, thanks to the rhyme:

\[ \text{Bear}(\text{FuzzyWuzzy}) \]

\[ \neg \text{HasHair}(\text{FuzzyWuzzy}) \]

(a) Write down the English-sentence equivalents of the six rules governing Fuzzy Wuzzy’s universe.

See above.

(b) Convert each of the six rules into Clausal Normal Form (CNF). Show your intermediate steps for each rule.

i. **Rule 1:** \( \{ \neg \text{Bear}(b), \text{Coat}(F(b)) \} \{ \neg \text{Bear}(b), \text{Has}(b, F(b)) \} \)

\[ \forall b \, \exists c \, \text{Bear}(b) \Rightarrow \text{Coat}(c) \land \text{Has}(b, c) \text{ (Original)} \]

\[ \forall b \, \exists c \, \neg \text{Bear}(b) \lor (\text{Coat}(c) \land \text{Has}(b, c)) \text{ (Eliminate implications)} \]

No need to move \( \neg \) inwards or standardize variables

\[ \forall b \, \neg \text{Bear}(b) \lor (\text{Coat}(F(b)) \land \text{Has}(b, F(b))) \text{ (Skolemize out } c \text{)} \]

\[ \neg \text{Bear}(b) \lor (\text{Coat}(F(b)) \land \neg \text{Has}(b, F(b))) \text{ (Drop universal quantifiers)} \]

\[ (\neg \text{Bear}(b) \lor \text{Coat}(F(b)) \land (\neg \text{Bear}(b) \lor \text{Has}(b, F(b)))) \text{ (Distribute } \lor \text{ over } \land). \]

ii. **Rule 2:** \( \{ \neg \text{Raincoat}(c), \neg \text{Furcoat}(c) \} \)

\[ \neg \exists c \, \text{Raincoat}(c) \land \text{Furcoat}(c) \text{ (Original)} \]

No implications to eliminate.

\[ \forall c \, \neg (\text{Raincoat}(c) \land \text{Furcoat}(c)) \text{ (Move } \neg \text{ inwards)} \]

\[ \forall c \, \neg \text{Raincoat}(c) \lor \neg \text{Furcoat}(c) \text{ (Move } \neg \text{ inwards)} \]

No need to standardize variables or skolemize.

\[ \neg \text{Raincoat}(c) \lor \neg \text{Furcoat}(c) \text{ (Drop universal quantifiers)} \]

No need to distribute \( \lor \) over \( \land \).

iii. **Rule 3:** \( \{ \neg \text{Coat}(c), \text{Raincoat}(c), \text{Furcoat}(c) \} \)

\[ \forall c \, \neg \text{Coat}(c) \Rightarrow (\text{Raincoat}(c) \lor \text{Furcoat}(c)) \text{ (Original)} \]

\[ \forall c \, \text{Coat}(c) \lor (\text{Raincoat}(c) \lor \text{Furcoat}(c)) \text{ (Eliminate implications)} \]

No need to move \( \neg \) inwards, standardize variables, or skolemize.

\[ \neg \text{Coat}(c) \lor (\text{Raincoat}(c) \lor \text{Furcoat}(c)) \text{ (Drop universal quantifiers)} \]

\[ \neg \text{Coat}(c) \lor \text{Raincoat}(c) \lor \text{Furcoat}(c) \text{ (Drop unnecessary parentheses)} \]

No need to distribute \( \lor \) over \( \land \).

iv. **Rule 4:** \( \{ \neg \text{Has}(x, c), \neg \text{Furcoat}(c), \text{Fuzzy}(x) \} \)

\[ \forall x, c \, \text{Has}(x, c) \land \text{Furcoat}(c) \Rightarrow \text{Fuzzy}(x) \text{ (Original)} \]
\(\forall x, c \neg(\text{Has}(x, c) \land \text{Furcoat}(c)) \lor \text{Fuzzy}(x)\) (Eliminate implications)

\(\forall x, c (\neg \text{Has}(x, c) \lor \neg \text{Furcoat}(c)) \lor \text{Fuzzy}(x)\) (Move \(\neg\) inwards)

\(\forall x, c \neg \text{Has}(x, c) \lor \neg \text{Furcoat}(c) \lor \text{Fuzzy}(x)\) (Drop unnecessary parentheses)

No need to standardize variables or skolemize.

\(\neg \text{Has}(x, c) \lor \neg \text{Furcoat}(c) \lor \text{Fuzzy}(x)\) (Drop universal quantifiers)

No need to distribute \(\lor\) over \(\land\).

v. **Rule 5**: \(\{\neg \text{HasHair}(x), \text{Fuzzy}(x)\}\)

\(\forall x \text{HasHair}(x) \Rightarrow \text{Fuzzy}(x)\) (Original)

\(\forall x \neg \text{HasHair}(x) \lor \text{Fuzzy}(x)\) (Eliminate implications)

No need to move \(\neg\) inwards, standardize variables, or skolemize.

\(\neg \text{HasHair}(x) \lor \text{Fuzzy}(x)\) (Drop universal quantifiers)

No need to distribute \(\lor\) over \(\land\).

vi. **Rule 6**: \(\{\neg \text{has}(\text{FuzzyWuzzy}, c), \neg \text{Raincoat}(c)\}\)

\(\neg \exists c \text{Has}(\text{FuzzyWuzzy}, c) \land \text{Raincoat}(c)\) (Original)

No need to eliminate implications.

\(\forall c \neg(\text{Has}(\text{FuzzyWuzzy}, c) \land \text{Raincoat}(c))\) (Move \(\neg\) inwards)

\(\forall c \neg \text{Has}(\text{FuzzyWuzzy}, c) \lor \neg \text{Raincoat}(c)\) (Move \(\neg\) inwards)

No need to standardize variables or skolemize.

\(\neg \text{Has}(\text{FuzzyWuzzy}, c) \lor \neg \text{Raincoat}(c)\) (Drop universal quantifiers)

No need to distribute \(\lor\) over \(\land\).

c) Using resolution on your clauses from part b), prove that Fuzzy Wuzzy is fuzzy. Number the steps in your resolution so that steps 1 – n are the n CNF clauses from part b) (plus the two clauses from the KB and the negated goal), and each subsequent clause is labeled with the numbers of the two clauses that you resolved to produce it. Also, if any unification was required for a particular step, write the substitution to the right of the resulting clause. For example:
1) $\neg\text{Bear}(b), \text{Coat}(F(b))$
2) $\neg\text{Bear}(b), \text{Has}(b, F(b))$
3) $\neg\text{Raincoat}(c), \neg\text{Furcoat}(c)$
4) $\neg\text{Coat}(c), \text{Raincoat}(c), \text{Furcoat}(c)$
5) $\neg\text{Has}(x, c), \neg\text{Furcoat}(c), \text{Fuzzy}(x)$
6) $\neg\text{HasHair}(x), \text{Fuzzy}(x)$
7) $\neg\text{Has}(\text{FuzzyWuzzy}, c), \neg\text{Raincoat}(c)$
8) $\text{Bear}($FuzzyWuzzy$)$
9) $\neg\text{HasHair}(\text{FuzzyWuzzy})$
10) $\neg\text{Fuzzy}($FuzzyWuzzy$)$
11) $\neg\text{Has}(\text{FuzzyWuzzy}, c), \neg\text{Furcoat}(c)$
12) $\neg\text{Coat}(c), \text{Raincoat}(c), \neg(\text{Has}(\text{FuzzyWuzzy}, c))$
13) $\neg\text{Bear}(b), \text{Raincoat}(F(b)), \neg\text{Has}(\text{FuzzyWuzzy}, F(b))$
14) $\text{Raincoat}(F(\text{FuzzyWuzzy})), \neg\text{Has}(\text{FuzzyWuzzy}, F(\text{FuzzyWuzzy}))$
15) $\neg\text{Has}(\text{FuzzyWuzzy}, F(\text{FuzzyWuzzy}))$
16) $\neg\text{Bear}(\text{FuzzyWuzzy})$
17) $\{}$

Clause 1 from Rule 1
Clause 2 from Rule 1
From Rule 2
From Rule 3
From Rule 4
From Rule 5
From Rule 6
From KB
From KB
Negated Goal
(10, 5) \{x/FuzzyWuzzy\}
(11, 4)
(12, 1) \{c/F(b)\}
(13, 8) \{b/FuzzyWuzzy\}
(14, 7) \{c/F(FuzzyWuzzy)\}
(15, 2) \{b/FuzzyWuzzy\}
(16, 8)