Constraint Satisfaction Problems

Chapter 3, Section 7 and Chapter 4, Section 4.4

Outline

- ◇ CSP examples
- ◇ General search applied to CSPs
- ◇ Backtracking
- ◇ Forward checking
- ◇ Heuristics for CSPs

Constraint satisfaction problems (CSPs)

Standard search problem:
state is a "black box"—any old data structure
that supports goal test, eval, successor

CSP:
state is defined by *variables* V_i with *values* from *domain* D_i

goal test is a set of *constraints* specifying
allowable combinations of values for subsets of variables

Simple example of a *formal representation language*

Allows useful *general-purpose* algorithms with more power than standard search algorithms

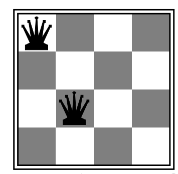
Example: 4-Queens as a CSP

Assume one queen in each column. Which row does each one go in?

Variables Q_1, Q_2, Q_3, Q_4

Domains $D_i = \{1, 2, 3, 4\}$

Constraints
$Q_i \neq Q_j$ (cannot be in same row)
$|Q_i - Q_j| \neq |i - j|$ (or same diagonal)



$Q_1 = 1$ $Q_2 = 3$

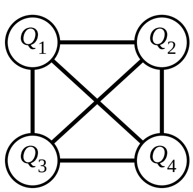
Translate each constraint into set of allowable values for its variables

E.g., values for (Q_1, Q_2) are $(1,3)$ $(1,4)$ $(2,4)$ $(3,1)$ $(4,1)$ $(4,2)$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints



Example: Cryptarithmetic

Variables
$D\ E\ M\ N\ O\ R\ S\ Y$

Domains
$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

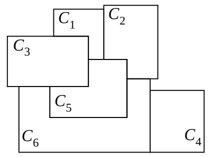
$$\begin{array}{rcccc} & S & E & N & D \\ + & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$

Constraints
$M \neq 0$, $S \neq 0$ (*unary* constraints)
$Y = D + E$ or $Y = D + E - 10$, etc.
$D \neq E$, $D \neq M$, $D \neq N$, etc.

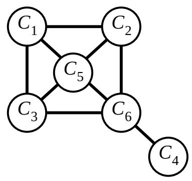
Example: Map coloring

Color a map so that no adjacent countries have the same color

Variables
  Countries C_i
Domains
  $\{Red, Blue, Green\}$
Constraints
  $C_1 \neq C_2$, $C_1 \neq C_5$, etc.



Constraint graph:



Real-world CSPs

Assignment problems
  e.g., who teaches what class

Timetabling problems
  e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

Applying standard search

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

Initial state: all variables unassigned

Operators: assign a value to an unassigned variable

Goal test: all variables assigned, no constraints violated

Notice that this is the same for all CSPs!

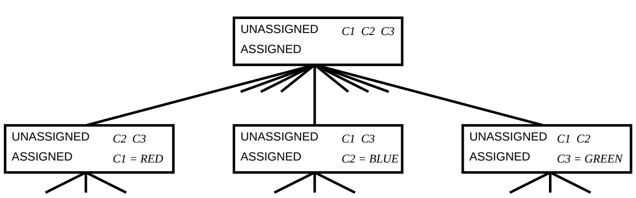
Implementation

CSP state keeps track of which variables have values so far
Each variable has a domain and a current value

datatype CSP-STATE
  components: UNASSIGNED, a list of variables not yet assigned
    ASSIGNED, a list of variables that have values
datatype CSP-VAR
  components: NAME, for i/o purposes
    DOMAIN, a list of possible values
    VALUE, current value (if any)

Constraints can be represented
  explicitly as sets of allowable values, or
  implicitly by a function that tests for satisfaction of the constraint

Standard search applied to map-coloring



Complexity of the dumb approach

Max. depth of space $m =$??

Depth of solution state $d =$??

Search algorithm to use??

Branching factor $b =$??

This can be improved dramatically by noting the following:

1) Order of assignment is irrelevant, hence many paths are equivalent
2) Adding assignments cannot correct a violated constraint

Complexity of the dumb approach

Max. depth of space $m =$?? n (number of variables)

Depth of solution state $d =$?? n (all vars assigned)

Search algorithm to use?? depth-first

Branching factor $b =$?? $\Sigma_i |D_i|$ (at top of tree)

This can be improved dramatically by noting the following:

1) Order of assignment is irrelevant so many paths are equivalent
2) Adding assignments cannot correct a violated constraint

Backtracking search

Use depth-first search, but
1) fix the order of assignment, $\Rightarrow b = |D_i|$
(can be done in the SUCCESSORS function)
2) check for constraint violations

The constraint violation check can be implemented in two ways:
1) modify SUCCESSORS to assign only values that
are allowed, given the values already assigned
or 2) check constraints are satisfied before expanding a state

Backtracking search is the basic uninformed algorithm for CSPs

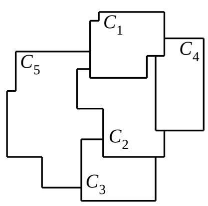
Can solve n-queens for $n \approx 15$

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

Simplified map-coloring example:

	RED	BLUE	GREEN
C_1			
C_2			
C_3			
C_4			
C_5			



Can solve n-queens up to $n \approx 30$

.
.
.

	$\surd$		
	$\times$		
	$\times$		
	$\times$		

.
.
.

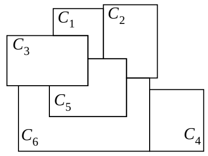
		$\surd$	
		$\times$	
		$\times$	
		$\times$	

.
.
.

			$\surd$
			$\times$

Heuristics for CSPs

More intelligent decisions on
- which value to choose for each variable
- which variable to assign next



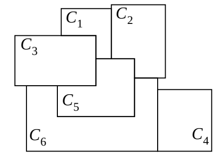
Given $C_1 = Red,\ C_2 = Green$, choose $C_3 =$??

Given $C_1 = Red,\ C_2 = Green$, what next??

Can solve n-queens for $n \approx 1000$

Heuristics for CSPs

More intelligent decisions on
- which value to choose for each variable
- which variable to assign next



Given $C_1 = Red,\ C_2 = Green$, choose $C_3 =$??
$C_3 = Green$: *least-constraining-value*
Given $C_1 = Red,\ C_2 = Green$, what next??
C_5: *most-constrained-variable*

Can solve n-queens for $n \approx 1000$

Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

To apply to CSPs:
- allow states with unsatisfied constraints
- operators *reassign* variable values

Variable selection: randomly select any conflicted variable

min-conflicts heuristic:
- choose value that violates the fewest constraints
- i.e., hillclimb with $h(n)$ = total number of violated constraints

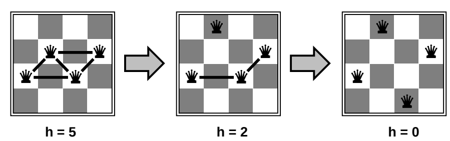
Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n)$ = number of attacks

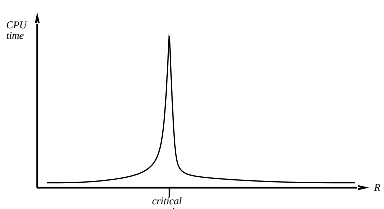


Performance of min-conflicts

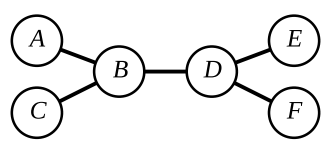
Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Tree-structured CSPs



Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n|D|^2)$ time

Compare to general CSPs, where worst-case time is $O(|D|^n)$

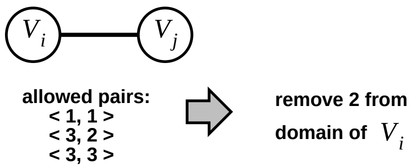
This property also applies to logical and probabilistic reasoning:
an important example of the relation between syntactic restrictions and complexity of reasoning.

Algorithm for tree-structured CSPs

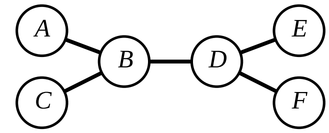
Basic step is called *filtering*:

$\textsc{Filter}(V_i, V_j)$
removes values of V_i that are inconsistent with ALL values of V_j

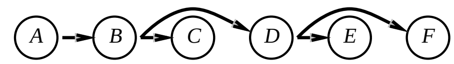
Filtering example:



Algorithm contd.



1) Order nodes breadth-first starting from any leaf:



2) For $j = n$ to 1, apply $\textsc{Filter}(V_i, V_j)$ where V_i is a parent of V_j

3) For $j = 1$ to n, pick legal value for V_j given parent value

Summary

CSPs are a special kind of problem:
states defined by values of a fixed set of variables
goal test defined by *constraints* on variable values

Backtracking = depth-first search with
1) fixed variable order
2) only legal successors

Forward checking prevents assignments that guarantee later failure

Variable ordering and value selection heuristics help significantly

Iterative min-conflicts is usually effective in practice

Tree-structured CSPs can always be solved very efficiently