Homework #1

Displays, Image Processing,

Affine Transformations, Hierarchical Modeling

Assigned: Friday, April 7th

Due: Thursday, April 20th
at the beginning of class

Directions: Please provide short written answers to the following questions on your own paper. Feel free to discuss the problems with classmates, but *please follow the Gilligan's Island rule*, answer the questions on your own, and show your work.*

Please write your name on your assignment!

* The Gilligan's Island Rule: This rule says that you are free to meet with fellow student(s) and discuss assignments with them. Writing on a board or shared piece of paper is acceptable during the meeting; however, you should not take any written (electronic or otherwise) record away from the meeting. After the meeting, engage in a half hour of mind-numbing activity (like watching an episode of Gilligan's Island), before starting to work on the assignment. This will assure that you are able to reconstruct what you learned from the meeting, by yourself, using your own brain.

Clarifications posted to class

Problem 2
- **Part (e)**: The sharpen parameter is beta. There was an alpha in the problem originally. The pdf was updated.

Problem 4
- The matrix M_xy was missing. (For some reason, MS Word had dropped it from the file. Grr.) Now fixed in the posted pdf.

Problem 5

- The pincer operates by **translating** primitives 7 and 8 toward or away from each other according to d.

- Assume that the center of part 1 is sitting at the origin in world coordinates. I updated the pdf to note this in part 5a, so that it applies to the rest of the problem.

- **Part (a):** You will need to apply scale factors to geometry. As stated in the problem, you can get these by measuring the figures with a ruler and assume that scale factors are either integers (1, 2, 3, ...) or simple fractions (1/2, 1/3, 1/4, ...). I update the intro above the figures to clarify this further, now in the homework pdf.

- **Part (a):** You should follow the structure that you see on slide 11 of the Hierarchical Modeling lecture. In that slide, the robot arm is rooted by the part "Base" and has an edge that connects it to its geometry "Cylinder" and another edge the connects to the rest of the model. "Base" is the same type as a "part number" in problem 5a (i.e., just a label), and "Cylinder" is the same as geometry (A or B) in this problem (i.e., the thing that actually gets drawn).

Problem 1: Short answer (10 points)

a) (2 Points) Suppose you removed the LCD panel in front of the unpolarized backlight – you removed the entire assembly with polarizers, crystals, color filters, everything – and looked right at the even backlighting with your naked eye(s); you would see even, white light. Roughly how much dimmer would you expect that light to become after putting the LCD panel back, assuming the framebuffer is set to white at each pixel for both eyes? [Note that unpolarized light intensity is cut in half by a linear polarizer. Assume that the R,G,B sub-pixels each transmit 1/3 of the visible spectrum of the light.] Justify your answer.

b) (2 points) What is the purpose of double-buffering and how does it work?

c) (2 points) Why does bilateral filtering not work for removing outlier noise (like salt-and-pepper or impulse noise)?

d) (4 Points) Consider two vectors **u** and **v** which are of non-zero length and not parallel to each other. Which of the following is true and which is false:

1. $\mathbf{v} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \times \mathbf{u}) \times \mathbf{v}$

2. $\mathbf{u} \cdot \left[\mathbf{v} \times (\mathbf{u} \times \mathbf{v}) \right] = 0$

3. $\mathbf{v} \cdot \left[\mathbf{v} \times (\mathbf{u} \times \mathbf{v}) \right] = 0$

4. $\mathbf{v} \cdot \dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \mathbf{u} \cdot \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

You do **not** need to justify your answer.

Problem 2: Image processing (25 points)

In this problem, you will consider several convolution filtering operations and their behaviors. You do not need to worry about flipping filters before sliding them across images; i.e., assume filters are pre-flipped. In addition, assume that the y-axis points up, the x-axis points to the right, and the lower left corner of the image is at (0,0). *For each sub-problem, justify your answer.*
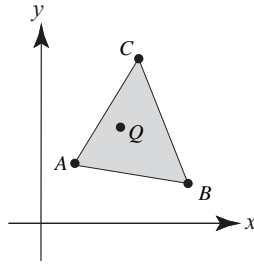
a) (3 points) The image you're editing is too dark and noisy, and you decide you need to blur the image a little with a 3x3 filter to reduce noise, and amplify the overall brightness of the image by a factor of 3. Suggest a single convolution filter that perform this task when applied to the image. (Technically, pixel values could go out of range, i.e., brighter than 255 in one or more color channels at a pixel; assume that any needed clamping will be taken care of later, after filtering).

b) (3 points) While taking a photograph with your digital camera, you fail to hold the camera steady, and it translates diagonally along the $x = y$ direction while the shutter is open. You discover this later when you see that diagonal $x = -y$ (or equivalently, $y = -x$) edges, in particular, have been blurred a bit (an effect called "motion blur"). You decide to filter the image so that diagonal $x = -y$ edges are sharpened, but diagonal $x = y$ edges are unchanged. Suggest a single convolution filter that does this.

c) (3 points) After thinking a little more about the previous picture, you decide that motion blur is cool, and you want to apply it to another image. In this case, though, you want to simulate the effect of a camera translating horizontally (in the x-direction) while the shutter is open. Suggest a convolution filter that would accomplish some horizontal blurring along that direction by averaging across m pixels.

d) (3 points) Describe a non-constant image that, when convolved with your horizontal blur filter from (c), would be unchanged by the filter. (You may ignore the boundaries.)

e) (7 points) We can think of sharpening as "removing blur" from an image. Suppose you take an image f and convolve it with a 3x3 mean filter. Now you multiply the resulting blurred image by β and subtract the result from the original image. You can think of the parameter β as controlling how much blur you remove. The resulting image will be a bit sharper, but also a bit darker, which can be corrected with suitable brightening. Solve for a single, normalized filter that performs all of these steps when convolved with the image, with β left as a variable in your filter. Note that the normalization will actually take care of the needed brightening. Show your work.

f) (2 points) What filter do you get from part (e) if you set $\beta = 1/2$? Try applying it using your Impressionist project. Does it sharpen?

g) (4 points) Suppose you follow the same sharpening process as in part (e), but this time you blur the original image with the following filter:

$$\begin{bmatrix} 0 & 1/5 & 0 \\ 1/5 & 1/5 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix}$$

then multiply the result by 5/6, subtract it from the original image, and fix the brightness. Solve for the (normalized) filter that will accomplish these steps all at once when convolved with the original image. Show your work. The result should resemble a filter we have discussed before — which one?

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Problem 3: Triangle coordinates (24 points)

Consider triangle Δ_{ABC} and a point Q depicted below:

A, B, C, and Q lie in the x-y plane, so, **neglecting the homogeneous coordinate**, we can write out their **3D** coordinates as:

$$A = \begin{bmatrix} A_x \\ A_y \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} B_x \\ B_y \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_x \\ Q_y \\ 0 \end{bmatrix}$$

The last coordinate is the z coordinate in this case, and we know that $A_z = B_z = C_z = Q_z = 0$. Note that Q is depicted as lying inside of Δ_{ABC}, but you should not assume that it does unless stated otherwise in a sub-problem. Assume that $A \neq B \neq C$, i.e., the triangle is not "degenerate." Further assume that if you curl the fingers of your right hand from A to B to C, your thumb will point in the direction of the positive z-axis.

(a) (3 points) Using cross and/or dot products, devise a test (with the help of equations) to determine if point Q lies inside of Δ_{ABC}. You should assume that the edges and vertices of the triangle are part of the interior of the triangle. [You do not need to expand any cross or dot products in your answer, but you may do so if it helps you.]

(b) (2 points) Using cross and/or dot products, compute the unit-length normal to Δ_{ABC}. Your solution should work for 3D points in general, i.e., not depend on the fact that these points lie in the x-y plane. We will use the right-hand rule for triangles, which means that as you curl the fingers of your right hand from A to B to C, your thumb will point in the direction of the normal. [You do not need to expand any cross or dot products in your answer, but you may do so if it helps you.]

(c) (3 points) Using cross and/or dot products, compute the area of the triangle, Area(Δ_{ABC}). This time you **do** need to expand all cross and/or dot products based on the elements of A, B, and C. Multiply out all terms; e.g., an expression like $(a+b)(c+d)$ should be expanded to $ac+ad+bc+bd$.

For the remainder of the problem, we will safely ignore the z coordinate and work in **2D** only, but **now we will keep track of the affine coordinate** ($w=1$ for all points). We can now represent the triangle vertices and the point Q as:

$$A = \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} C_x \\ C_y \\ 1 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix}$$

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Again, to be clear, the last coordinate is now the affine w coordinate, which, for affine points, is always 1; i.e., $A_w = B_w = C_w = Q_w = 1$.

(d) (2 points) Suppose we create a 3x3 matrix $[A\ B\ C]$, i.e., a matrix with columns filled by A, B, and C. Write out this matrix, explicitly filling in the elements of the matrix in terms of the elements of A, B, and C, and compute its determinant, $\det[A\ B\ C]$. Again, multiply out all terms.

(e) (1 point) Based on your answers to (c) and (d), what is Area($\triangle ABC$) in terms of $\det[A\ B\ C]$?

(f) (1 point) In general, we can express Q as a weighted sum of A, B, and C; i.e., $Q = \alpha A + \beta B + \gamma C$, where α, β, and γ are scalars. In order for this to be a proper affine combination (a weighted sum of affine points that yields an affine point), what constraint is placed on α, β, and γ? Explain.

(g) (3 points) Now we will work on solving for α, β, and γ. Write out a matrix equation of the form $M\mathbf{p} = \mathbf{r}$:

$$
\begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}
$$

where M is a 3x3 matrix, $\mathbf{p}$ is the column vector of unknowns, i.e., $\mathbf{p} = [\alpha\ \beta\ \gamma]^{\mathrm{T}}$ and $\mathbf{r}$ is a column vector with three elements. I.e., explicitly write out the matrix equation, filling in the elements of M, $\mathbf{p}$, and $\mathbf{r}$, in terms of α, β, γ, and the elements of A, B, C, and Q. (Do not apply the matrix, just set up the equation.) Hint: you can expand $Q = \alpha A + \beta B + \gamma C$ explicitly in terms of the elements of A, B, C, and Q and the result should be equivalent to your matrix equation.

(h) (2 points) We can solve for $\mathbf{p}$ using Cramer's rule. In particular, for a matrix equation $M\mathbf{p} = \mathbf{r}$ as above, we can solve for $\mathbf{p}$ as ratios of determinants:

$$
p_0 = \frac{\det \begin{bmatrix} r_0 & m_{01} & m_{02} \\ r_1 & m_{11} & m_{12} \\ r_2 & m_{21} & m_{22} \end{bmatrix}}{\det \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}} \quad
p_1 = \frac{\det \begin{bmatrix} m_{00} & r_0 & m_{02} \\ m_{10} & r_1 & m_{12} \\ m_{20} & r_2 & m_{22} \end{bmatrix}}{\det \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}} \quad
p_2 = \frac{\det \begin{bmatrix} m_{00} & m_{01} & r_0 \\ m_{10} & m_{11} & r_1 \\ m_{20} & m_{21} & r_2 \end{bmatrix}}{\det \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}}
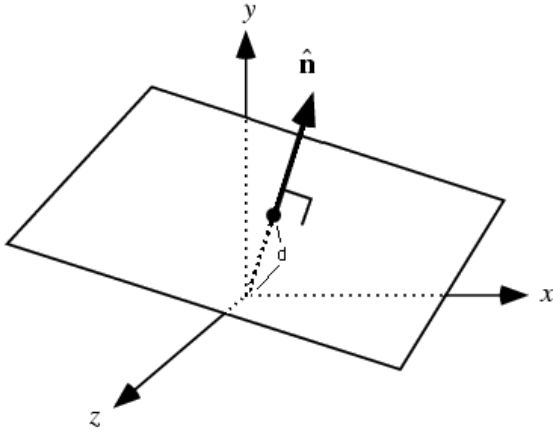$$

Note how the denominator is always the determinant of M, and the numerator is the determinant of a matrix that consists of M with one of the columns replaced with the elements of $\mathbf{r}$. Based on your answer to (g), re-write these same "Cramer's rule" equations using α, β, γ, and the elements of A, B, C, and Q.

(i) (3 points) Assume Q is inside of $\triangle ABC$. In this case, all of the determinants in (h) are positive. Based on your answer to (e), re-write your answer to (h) in terms of areas of triangles.

(j) (2 points) Suppose $Q = B$. What should α, β, and γ be? Justify your answer in terms of areas of triangles.

(k) (2 points) Suppose Q is halfway between A and C. What should α, β, and γ be? Justify your answer in terms of areas of triangles.

Problem 4. 3D affine transformations (15 points)

The equation $\hat{\mathbf{n}} \bullet \bar{\mathbf{x}} = d$ describes the plane pictured below which has unit length normal $\hat{\mathbf{n}}$ pointing away from the origin and is a distance d from the origin (in the direction of the normal vector). Any point $\bar{\mathbf{x}} = \begin{bmatrix} x & y & z \end{bmatrix}$ on the plane must satisfy the plane equation $\hat{\mathbf{n}} \bullet \bar{\mathbf{x}} = d$.
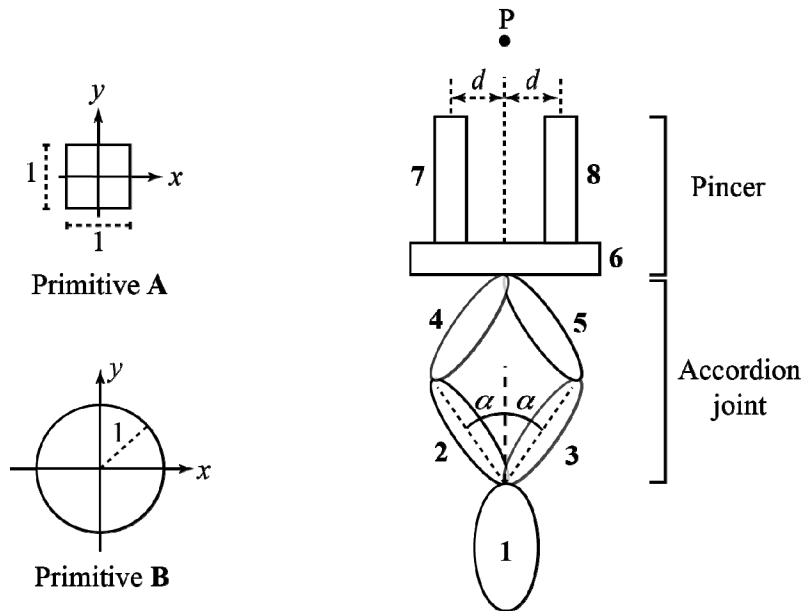

$$M_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now consider a plane with normal lying in the y-z plane. The normal will have the form $(0, \sin\theta, \cos\theta)$ for some θ. The equation for the plane is then $y \sin\theta + z \cos\theta = d$. Write out the product of 4x4 matrices that would perform a reflection across this plane. One of these matrices will be a reflection matrix; you must use the matrix M_{xy} above, which performs a reflection across the x-y plane. You must write out the elements of the matrices and the product order in which they would be applied, but you do not need to multiply them out. Justify your answer with words and/or drawings.

Problem 5. Hierarchical modeling (26 points)

Suppose you want to model a pincer with accordion joint illustrated below. The model is comprised of 8 parts, using primitives **A** and **B**. The model is controlled by parameters α and d. The pincer operates by translating primitives 7 and 8 toward or away from each other according to d. The illustration on the right also shows a point P that the model is reaching toward, as described in sub-problem **c**).

Assume that α can take values in the range $[0, 90°]$. As part of your solution to this problem, you will need to define the instance transformations that should be applied to a given primitive on the left so that it is the same shape as a desired primitive on the right; all scale factors will be integers (1, 2, 3, …) or simple fractions (1/2, 1/3, 1/4, …); use a ruler to determine scale factors. Transformations available to you:

- $R(\theta)$ – rotate by θ degrees (counter clockwise)

- $T(a, b)$ – translate by $\begin{bmatrix} a \\ b \end{bmatrix}$

- $S(s_x, s_y)$ – scale the x-component by s_x and scale the y-component by s_y

a) (20 points) Construct a **tree** to describe this hierarchical model using part **1** as the root. Each node will contain either a part number (**1**…**8**) or a reference to the canonical geometry that will be drawn (**A** or **B**). Along each of the edges of the tree, write an expression for the transformations that are applied along that edge to connect parent to child. Write all transformations using the notation above; you do not need to write out the matrices, just the symbolic references to them, and their arguments. Remember that the order of transformations is important! Show your work wherever the transformations are not "obvious." Assume that the *center* of part **1** sits on the origin in world coordinates. The tree must have one or more branches in it. If two parts are connected physically, then they should be connected in the tree, as long as you don't end up with any node with two parents.

b) (2 points) Write out the full transformation expression for drawing the geometric primitive for part **7**. Again use the symbolic matrix notation above that appears in your tree.

c) (4 points) What values of α and d would have the model extend out and close the pincer just enough to precisely grasp the point $P = [0\ \ 6]^T$. Show your work.

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