Mathematical surface representations

- Explicit: $z = f(x,y)$ (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?

- Implicit: $g(x,y,z) = 0$

- Parametric: $S(u,v) = (x(u,v), y(u,v), z(u,v))$
  - For the sphere:
    \[ x(u,v) = r \cos 2\pi v \sin \pi u \]
    \[ y(u,v) = r \sin 2\pi v \sin \pi u \]
    \[ z(u,v) = r \cos \pi u \]

As with curves, we'll focus on parametric surfaces.

Surfaces of revolution

Recall that surfaces of revolution are based on the idea of rotating about an axis…

Given: A set of points $C[n]$ on a curve in the $xy$-plane:

\[ C[n] = \begin{bmatrix} C_1[n] \\ C_2[n] \\ 0 \\ 1 \end{bmatrix} \quad \text{where } n \in [0, N] \]

Let $\theta_m$ be a rotation about the $y$-axis by angle $\theta_m$.

Find: A set of points $S[n,m]$ on the surface formed by rotating $C[n]$ rotated about the $y$-axis. Assume $m \in [0, M]$.

Solution:

\[ S[n,m] = R_m[n] \cdot C[n] \]
General sweep surfaces

The surface of revolution is a special case of a swept surface.

Idea: Trace out surface $S(u,v)$ by moving a profile curve $C(u)$ along a trajectory curve $T(v)$.

More specifically:
- Suppose that $C(u)$ lies in an $(x_c,y_c)$ coordinate system with origin $O_c$.
- For every point along $T(v)$, lay $C(u)$ so that $O_c$ coincides with $T(v)$.

Orientation

The big issue:
- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:
1. Fixed (or static): Just translate $O_c$ along $T(v)$.
2. Moving. Use the Frenet frame of $T(v)$.
   - Allows smoothly varying orientation.
   - Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.

To get a 3D coordinate system, we need 3 independent direction vectors.
- Tangent: $t(v) = \text{normalize}[T'(v)]$
- Binormal: $b(v) = \text{normalize}[T'(v) \times T''(v)]$
- Normal: $n(v) = b(v) \times t(v)$

As we move along $T(v)$, the Frenet frame $(t,b,n)$ varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:
- Put $C(u)$ in the normal plane.
- Place $O_c$ on $T(v)$.
- Align $x_c$ for $C(u)$ with $b$.
- Align $y_c$ for $C(u)$ with $-n$.

If $T(v)$ is a circle, you get a surface of revolution exactly!
Degenerate frames

Let's look back at where we computed the coordinate frames from curve derivatives:

Where might these frames be ambiguous or undetermined?

Variations

Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\hat{C}(u)$ as it moves along $T(v)$.
- ...

Tensor product Bézier surfaces

Given a grid of control points $V_{ij}$ forming a control net, construct a surface $S(u,v)$ by:

- treating rows of $V$ (the matrix consisting of the $V_{ij}$) as control points for curves $V_0(u)$,..., $V_n(u)$.
- treating $V_0(u)$,..., $V_n(u)$ as control points for a curve parameterized by $v$.

Tensor product Bézier surfaces, cont.

Let's walk through the steps:

Which control points are interpolated by the surface?
Polynomial form of Bézier surfaces

Recall that cubic Bézier curves can be written in terms of the Bernstein polynomials:

\[ Q(u) = \sum_{i=0}^{3} V_i b_i(u) \]

A tensor product Bézier surface can be written as:

\[ S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{n} V_{ij} b_i(u) b_j(v) \]

In the previous slide, we constructed curves along \( u \), and then along \( v \). This corresponds to re-grouping the terms like so:

\[ S(u, v) = \sum_{i=0}^{n} \left( \sum_{j=0}^{n} V_{ij} b_i(u) \right) b_j(v) \]

But, we could have constructed them along \( v \), then \( u \):

\[ S(u, v) = \sum_{j=0}^{n} \left( \sum_{i=0}^{n} V_{ij} b_i(v) \right) b_j(u) \]

Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce \( C^2 \) continuity and local control, we get B-spline curves:

- treat rows of \( B \) as control points to generate Bézier control points in \( u \).
- treat Bézier control points in \( u \) as B-spline control points in \( v \).
- treat B-spline control points in \( v \) to generate Bézier control points in \( u \).

Tensor product B-spline surfaces, cont.

Another example:

Which B-spline control points are interpolated by the surface?
NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:

We can do this by trimming the u-v domain.

- Define a closed curve in the u-v domain (a trim curve)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Summary

What to take home:
- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces