Mathematical surface representations

- Explicit: $z = f(x,y)$ (a.k.a., a “height field”)
  - what if the curve isn’t a function, like a sphere?
    $x^2 + y^2 + z^2 = r^2$

- Implicit: $g(x,y,z) = 0$
  $g(x,y,z) = x^2 + y^2 + z^2 - r^2$

- Parametric: $S(u,v) = (x(u,v), y(u,v), z(u,v))$
  - For the sphere:
    $x(u,v) = r \cos 2\pi u \sin \pi v$
    $y(u,v) = r \sin 2\pi u \sin \pi v$
    $z(u,v) = r \cos \pi v$

As with curves, we'll focus on parametric surfaces.

Surfaces of revolution

Idea: rotate a 2D profile curve around an axis.

What kinds of shapes can you model this way?
Constructing surfaces of revolution

Given: A curve \( C(u) \) in the \( xy \)-plane:

\[
C(u) = \begin{bmatrix}
    c(u) \\
    s(u) \\
    0
\end{bmatrix}
\]

Let \( R_y(\theta) \) be a rotation about the \( y \)-axis.

Fixed: A surface \( S(u,v) \) which is \( C(u) \) rotated about the \( y \)-axis.

Solution:

Isoparameter curves and tangents

We can follow curves where \( v \) is constant, and \( u \) varies or vice versa. These are called isoparameter curves (where one parameter is held constant):

If we sample at equal spacing in \( u \) and \( v \), we can create a quadrilateral mesh (or a triangle mesh).

We can compute tangents to the surface at any point by looking at (infinitesimally) nearby points.

Holding one parameter constant, we can find nearby points by varying the other parameter.

Thus, we can get two tangents:

\[
t_u = \frac{\partial C(u,v)}{\partial u} \\
t_v = \frac{\partial C(u,v)}{\partial v}
\]

How would we compute the normal?

General sweep surfaces

The surface of revolution is a special case of a swept surface.

Idea: Trace out surface \( S(u,v) \) by moving a profile curve \( C(u) \) along a trajectory curve \( T(v) \).

More specifically:

- Suppose that \( C(u) \) lies in an \( (x_c, y_c) \) coordinate system with origin \( O_c \).
- For every point along \( T(v) \), lay \( C(u) \) so that \( O_c \) coincides with \( T(v) \).

Orientation

The big issue:

- How to orient \( C(u) \) as it moves along \( T(v) \)?

Here are two options:

1. **Fixed (or static):** Just translate \( O_c \) along \( T(v) \).
2. **Moving.** Use the Frenet frame of \( T(v) \).

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.
Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.

To get a 3D coordinate system, we need 3 independent direction vectors.
- **Tangent:** $t(v) = \text{normalize}(T'(v))$
- **Bi-normal:** $b(v) = \text{normalize}(T'(v) \times T''(v))$
- **Normal:** $n(v) = t(v) \times b(v)$

As we move along $T(v)$, the Frenet frame $(t,b,n)$ varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:
- Put $C(u)$ in the normal plane.
- Place $O_c$ on $T(v)$.
- Align $x_c$ for $C(u)$ with $b$.
- Align $y_c$ for $C(u)$ with $-n$.

If $T(v)$ is a circle, you get a surface of revolution exactly!

Degenerate frames

Let’s look back at where we computed the coordinate frames from curve derivatives:

Where might these frames be ambiguous or undetermined?

Variations

Several variations are possible:
- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- ...
Given a grid of control points $V_{ij}$ forming a control net, construct a surface $S(u,v)$ by:

- treating rows of $V$ (the matrix consisting of the $V_{ij}$) as control points for curves $V_0(u), \ldots, V_n(u)$.
- treating $V(u), \ldots, V(u)$ as control points for a curve parameterized by $v$.

Tensor product Bézier surfaces, cont.

Let’s walk through the steps:

Which control points are interpolated by the surface?

Polynomial form of Bézier surfaces

Recall that cubic Bézier curves can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^{3} v_i b_i(u)$$

A tensor product Bézier surface can be written as:

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} v_{ij} b_{ij}(u,v)$$

In the previous slide, we constructed curves along $u$, and then along $v$. This corresponds to re-grouping the terms like so:

$$S(u,v) = \sum_{i=0}^{n} \left( \sum_{j=0}^{m} v_{ij} b_{ij}(u) \right) b_i(v)$$

But, we could have constructed them along $v$, then $u$:

$$S(u,v) = \sum_{j=0}^{m} \left( \sum_{i=0}^{n} v_{ij} b_{ij}(v) \right) b_j(u)$$

Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce $C^2$ continuity and local control, we get B-spline curves:

- treat rows of $B$ as control points to generate Bézier control points in $u$.
- treat Bézier control points in $u$ as B-spline control points in $v$.
- treat B-spline control points in $v$ to generate Bézier control points in $u$. 
Tensor product B-spline surfaces, cont.

Which B-spline control points are interpolated by the surface?

Another example:

NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:

We can do this by trimming the u-v domain.

- Define a closed curve in the u-v domain (a trim curve)
- Do not draw the surface points inside of this curve.

It’s really hard to maintain continuity in these regions, especially while animating.
Summary

What to take home:

- How to construct a surface of revolution
- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces