Parametric curves

Curves before computers

The “loftsman’s spline”:
- long, narrow strip of wood or metal
- shaped by lead weights called “ducks”
- gives curves with second-order continuity, usually

Used for designing cars, ships, airplanes, etc.

But curves based on physical artifacts can’t be replicated well, since there’s no exact definition of what the curve is.

Around 1960, a lot of industrial designers were working on this problem.

Today, curves are easy to manipulate on a computer and are used for CAD, art, animation, …

Mathematical curve representation

- Explicit  \( y = f(x) \)
  - what if the curve isn’t a function, e.g., a circle?

- Implicit  \( g(x,y) = 0 \)

- Parametric  \( (x(u),y(u)) \)
  - For the circle:
    \[
    x(u) = \cos 2\pi u \\
    y(u) = \sin 2\pi u 
    \]
Parametric polynomial curves

We'll use parametric curves, \( Q(u) = (x(u), y(u)) \), where the functions are all polynomials in the parameter.

\[
x(u) = \sum_{k=0}^{n} a_k u^k
\]
\[
y(u) = \sum_{k=0}^{n} b_k u^k
\]

Advantages:
- easy (and efficient) to compute
- infinitely differentiable

We'll also assume that \( u \) varies from 0 to 1.

de Casteljau’s algorithm

Recursive interpolation:

What if \( u = 0 \)?

What if \( u = 1 \)?

de Casteljau’s algorithm, cont’d

Recursive notation:

What is the equation for \( V'_0 \)?

Finding \( Q(u) \)

Let's solve for \( Q(u) \):

\[
V'_0 = (1-u)V_0 + uV_1
\]
\[
V'_1 = (1-u)V'_0 + uV_2
\]
\[
V'_2 = (1-u)V'_1 + uV_3
\]
\[
Q(u) = (1-u)V'_2 + uV_3
\]

\[
= (1-u)[(1-u)V'_0 + uV'_1] + u[(1-u)V'_1 + uV'_2]
\]
\[
= (1-u)[(1-u)(V_0 + uV_1) + u(V'_0 + uV'_1)] + ... 
\]
\[
= (1-u)^3V_0 + 3u(1-u)^2V_1 + 3u^2(1-u)V_2 + u^3V_3
\]
Finding $Q(u)$ (cont’d)

In general,

$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} V_i$$

where $"n \text{ choose } i"$ is:

$$\binom{n}{i} = \frac{n!}{(n-i)!i!}$$

This defines a class of curves called Bézier curves.

What’s the relationship between the number of control points and the degree of the polynomials?

Bernstein polynomials

The coefficients of the control points are a set of functions called the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^{n} b_i(u) V_i$$

For degree 3, we have:

$$b_0(u) = (1-u)^3$$

$$b_1(u) = 3u(1-u)^2$$

$$b_2(u) = 3u^2(1-u)$$

$$b_3(u) = u^3$$

Useful properties on the interval $[0,1]$:

- each polynomial has value between 0 and 1
- sum of all four is exactly 1 (a.k.a., a “partition of unity”)

These together imply that the curve is generated by convex combinations of the control points and therefore lies within the convex hull of those control points.

Displaying Bézier curves

How could we draw one of these things?

It would be nice if we had an adaptive algorithm, that would take into account flatness.

```plaintext
DisplayBez( V0, V1, V2, V3 )
begin
  if ( FlatEnough( V0, V1, V2, V3 ) )
    Line( V0, V3 );
  else
    something;
end;
```

Subdivide and conquer

```plaintext
DisplayBez( V0, V1, V2, V3 )
begin
  if ( FlatEnough( V0, V1, V2, V3 ) )
    Line( V0, V3 );
  else
    Subdivide( V[ ] ) \Rightarrow L[ ], R[ ]
    DisplayBez( L0, L1, L2, L3 );
    DisplayBez( R0, R1, R2, R3 );
end;
```
Testing for flatness

Compare total length of control polygon to length of line connecting endpoints:

\[ |V_0 - V_1| + |V_1 - V_2| + |V_2 - V_3| < 1 + \varepsilon \]

Curve desiderata

Bézier curves offer a fairly simple way to model parametric curves.

But, let’s consider some general properties we would like curves to have…

Local control

One problem with Béziers is that every control point affects every point on the curve (except the endpoints).

Moving a single control point affects the whole curve!

We’d like to have local control, that is, have each control point affect some well-defined neighborhood around that point.

Interpolation

Bézier curves are approximating. The curve does not (necessarily) pass through all the control points. Each point pulls the curve toward it, but other points are pulling as well.

We’d like to have a curve that is interpolating, that is, that always passes through every control point.
Continuity

We want our curve to have **continuity**: there shouldn’t be any abrupt changes as we move along the curve.

"0th order" continuity would mean that curve doesn’t jump from one place to another.

We can also look at derivatives of the curve to get higher order continuity.

1st and 2nd Derivative Continuity

First order continuity implies continuous first derivative:

\[ Q'(u) = \frac{dQ(u)}{du} \]

Let’s think of \( u \) as “time” and \( Q(u) \) as the path of a particle through space. What is the meaning of the first derivative, and which way does it point?

Second order continuity means continuous second derivative:

\[ Q''(u) = \frac{d^2Q(u)}{du^2} \]

What is the intuitive meaning of this derivative?

C\( n \) (Parametric) Continuity

In general, we define C\( n \) continuity as follows:

\( Q(u) \) is C\( n \) continuous iff \( Q^{(i)}(u) = \frac{d^i Q(u)}{du^i} \) is continuous for \( 0 \leq i \leq n \)

Note: these are nested degrees of continuity:

\( C^{-1} \):

\( C^0 \):

\( C^1, C^2 \):

\( C^3, C^4, ... \):

Reparameterization

We have so far been considering parametric continuity, derivatives w.r.t. the parameter \( u \).

This form of continuity makes sense particularly if we really are describing a particle moving over time and want its motion (e.g., velocity and acceleration) to be smooth.

But, what if we’re thinking only in terms of the shape of the curve? Is the parameterization actually intrinsic to the shape, i.e., is it the case that a shape has only one parameterization?
Arc length parameterization

We can reparameterize a curve so that equal steps in parameter space (we’ll call this new parameter “s”) map to equal distances along the curve:

\[ Q(s) \Rightarrow \Delta s = s_2 - s_1 = \text{arclength}[Q(s_1), Q(s_2)] \]

We call this an arc length parameterization. We can re-write the equal step requirement as:

\[ \frac{\text{arclength}[Q(s_1), Q(s_2)]}{s_2 - s_1} = 1 \]

Looking at very small steps, we find:

\[ \lim_{s_2 \to s_1} \frac{\text{arclength}[Q(s_1), Q(s_2)]}{s_2 - s_1} = \left\| \frac{dQ(s)}{ds} \right\| = 1 \]

\[ G^n (Geometric) Continuity \]

Now, we define geometric \( G^n \) continuity as follows:

\[ Q(s) \text{ is } G^n \text{ continuous iff } \]

\[ \frac{d^{i}Q(s)}{ds^{i}} \text{ is continuous for } 0 \leq i \leq n \]

Where \( Q(s) \) is parameterized by arc length.

The first derivative still points along the tangent, but its length is always 1.

\( G^n \) continuity is usually a weaker constraint than \( C^n \) continuity (e.g., “speed” along the curve does not matter).

\[ \text{G}^n \text{ Continuity (cont’d)} \]

The second derivative now has a specific geometric interpretation. First, the “osculating circle” at a point on a curve can be defined based on the limit behavior of three points moving toward each other:

\[ O(s_1, s_2, s_3) \]

\[ Q(s_1), Q(s_2), Q(s_3) \]

\[ O(s) = \lim_{s_1, s_2, s_3 \to s} O(s_1, s_2, s_3) \]

The second derivative \( Q''(s) \) then has these properties:

\[ \left\| Q''(s) \right\| = \kappa(s) = \frac{1}{r(s)} = Q''(s) - c(s) - Q(s) \]

where \( r(s) \) and \( c(s) \) are the radius and center of \( O(s) \), respectively, and \( \kappa(s) \) is the “curvature” of the curve at \( s \).

We’ll focus on \( C^n \) (i.e., parametric) continuity of curves for the remainder of this lecture.

\[ \text{Bézier curves} \rightarrow \text{splines} \]

Bézier curves have C-infinity continuity on their interiors, but we saw that they do not exhibit local control or interpolate their control points.

It is possible to define points that we want to interpolate, and then solve for the Bézier control points that will do the job.

But, you will need as many control points as interpolated points -> high order polynomials -> wiggly curves. (And you still won’t have local control.)

Instead, we’ll splice together a curve from individual Béziers segments, in particular, cubic Béziers.

We call these curves splines.

The primary concern when splicing curves together is getting good continuity at the endpoints where they meet…
Ensuring $C^0$ continuity

Suppose we have a cubic Bézier defined by $(V_0, V_1, V_2, V_3)$, and we want to attach another curve $(W_0, W_1, W_2, W_3)$ to it, so that there is $C^0$ continuity at the joint.

\[ C^0 : Q_x(1) = Q_y(0) \]

What constraint(s) does this place on $(W_0, W_1, W_2, W_3)$?

The $C^0$ Bezier spline

How then could we construct a curve passing through a set of points $P_1 \ldots P_n$?

We call this curve a spline. The endpoints of the Bezier segments are called joints.

In the animator project, you will construct such a curve by specifying all the Bezier control points directly.

1st derivatives at the endpoints

For degree 3 (cubic) curves, we have already shown that we get:

\[ Q(u) = (1-u)^3V_0 + 3u(1-u)^2V_1 + 3u^2(1-u)V_2 + u^3V_3 \]

We can expand the terms in $u$ and rearrange to get:

\[ Q(u) = (-V_0 + 3V_1 - 3V_2 + V_3)u^3 + (3V_0 - 6V_1 + 3V_2)u^2 + (-3V_0 + 3V_1)u + V_0 \]

What then is the first derivative when evaluated at each endpoint, $u=0$ and $u=1$?

\[ Q'(0) = \]

\[ Q'(1) = \]

Ensuring $C^1$ continuity

Suppose we have a cubic Bézier defined by $(V_0, V_1, V_2, V_3)$, and we want to attach another curve $(W_0, W_1, W_2, W_3)$ to it, so that there is $C^1$ continuity at the joint.

\[ C^1 : \begin{cases} Q_x(1) = Q_y(0) \\ Q_x'(1) = Q_y'(0) \end{cases} \]

What constraint(s) does this place on $(W_0, W_1, W_2, W_3)$?
The $C^1$ Bezier spline

How then could we construct a curve passing through a set of points $P_0 \ldots P_n$?

We can specify the Bezier control points directly, or we can devise a scheme for placing them automatically...

Catmull-Rom splines

If we set each derivative to be one half of the vector between the previous and next controls, we get a Catmull-Rom spline.

This leads to:

\[
\begin{align*}
V_0 &= P_1 \\
V_1' &= P_1 + \frac{1}{6}(P_2 - P_0) \\
V_2' &= P_2 - \frac{1}{6}(P_3 - P_1) \\
V_3' &= P_2
\end{align*}
\]

We can specify the Bezier control points directly, or we can devise a scheme for placing them automatically...

Catmull-Rom to Beziers

We can write the Catmull-Rom to Bezier transformation as:

\[
\begin{bmatrix}
V_0^T \\
V_1^T \\
V_2^T \\
V_3^T
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1/6 & 1 & 1/6 & 0 \\
0 & 1/6 & 1 & -1/6 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
P_0^T \\
P_1^T \\
P_2^T \\
P_3^T
\end{bmatrix}
\]

\[
V = M_{\text{Catmull-Rom}} P
\]

Endpoints of Catmull-Rom splines

We can see that Catmull-Rom splines don't interpolate the first and last control points.

By repeating those control points, we can force interpolation.
**Tension control**

We can give more control by exposing the derivative scale factor as a parameter:

\[
\begin{align*}
V_0 &= P_1 \\
V_1 &= P_1 + \frac{1}{2}(P_2 - P_0) \\
V_2 &= P_2 - \frac{1}{2}(P_3 - P_1) \\
V_3 &= P_3
\end{align*}
\]

The parameter \( \tau \) controls the tension. Catmull-Rom uses \( \tau = 1/2 \). Here’s an example with \( \tau = 3/2 \).

**2nd derivatives at the endpoints**

Finally, we’ll want to develop \( C^2 \) splines. To do this, we’ll need second derivatives of Bezier curves.

Taking the second derivative of \( Q(u) \) yields:

\[
\begin{align*}
Q''(0) &= 6(V_0 - 2V_1 + V_2) \\
&= -6[(V_1 - V_0) + (V_2 - V_1)] \\
Q''(1) &= 6(V_1 - 2V_2 + V_3) \\
&= -6[(V_2 - V_1) + (V_3 - V_2)]
\end{align*}
\]

**Ensuring \( C^2 \) continuity**

Suppose we have a cubic Bézier defined by \( (V_0, V_1, V_2, V_3) \), and we want to attach another curve \( (W_0, W_1, W_2, W_3) \) to it, so that there is \( C^2 \) continuity at the joint.

\[
\begin{align*}
Q_1'(0) &= Q_0'(0) \\
Q_1'(1) &= Q_0'(0) \\
Q_1'(0) &= Q_0'(1)
\end{align*}
\]

What constraint(s) does this place on \( (W_0, W_1, W_2, W_3) \)?

**Building a complex spline**

Instead of specifying the Bézier control points themselves, let’s specify the corners of the A-frames in order to build a \( C^2 \) continuous spline.

These are called **B-splines**. The starting set of points are called **de Boor points**.
**B-splines**

Here is the completed B-spline.

What are the Bézier control points, in terms of the de Boor points?

\[
V_0 = \[\ldots \, B_0 + \ldots B_1 \]
+ \[\ldots \, B_1 + \ldots B_2 \]
= \[\ldots B_0 + \ldots B_1 + \ldots B_2 \]

\[
V_1 = \ldots B_1 + \ldots B_2
\]

\[
V_2 = \ldots B_1 + \ldots B_2
\]

\[
V_3 = \ldots B_1 + \ldots B_2 + \ldots B_3
\]

**B-splines to Beziers**

We can write the B-spline to Bezier transformation as:

\[
\begin{bmatrix}
V_0' \\
V_1' \\
V_2' \\
V_3'
\end{bmatrix} = \begin{bmatrix}
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
0 & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & \frac{2}{3} & 0 \\
0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{bmatrix} \begin{bmatrix}
B_0' \\
B_1' \\
B_2' \\
B_3'
\end{bmatrix}
\]

\[
V = M_{B-spline} B
\]

**Endpoints of B-splines**

As with Catmull-Rom splines, the first and last control points of B-splines are generally not interpolated.

Again, we can force interpolation by repeating the endpoints…twice.

**Closing the loop**

What if we want a closed curve, i.e., a loop?

With Catmull-Rom and B-spline curves, this is easy:
Curves in the animator project

In the animator project, you will draw a curve on the screen:

\[ Q(u) = (x(u), y(u)) \]

You will actually treat this curve as:

\[ \theta(u) = y(u) \]
\[ t(u) = x(u) \]

Where \( \theta \) is a variable you want to animate. We can think of the result as a function:

\[ \theta(t) \]

In general, you have to apply some constraints to make sure that \( \theta(t) \) actually is a function.

Summary

What to take home from this lecture:

- Geometric and algebraic definitions of Bézier curves.
- Basic properties of Bézier curves.
- How to display Bézier curves with line segments.
- Meanings of \( C^k \) continuities.
- Geometric conditions for continuity of cubic splines.
- Properties of B-splines and Catmull-Rom splines.
- Geometric construction of B-splines and Catmull-Rom splines.
- How to construct closed loop splines.