What is an image?

We can think of an image as a function, \( f \), from \( \mathbb{R}^2 \) to \( \mathbb{R} \):

- \( f(x, y) \) gives the intensity of a channel at position \((x, y)\)
- Realistically, we expect the image only to be defined over a rectangle, with a finite range:
  \[ f(x,y) : [a,b] \times [c,d] \rightarrow [0,1] \]

A color image is just three functions pasted together. We can write this as a “vector-valued” function:

\[
f(x, y) = \begin{bmatrix} \rho(x,y) \\ \gamma(x,y) \\ \delta(x,y) \end{bmatrix}
\]
What is a digital image?

In computer graphics, we usually operate on digital (discrete) images.

- Sample the space on a regular grid
- Quantize each sample (round to nearest integer)

If our samples are $\Delta$ apart, we can write this as:

$$f(x,y) \rightarrow \text{Quantized}(f(x,y) / \Delta)$$

Pixel movement

Some operations preserve intensities, but move pixels around in the image

$$g(x,y) = f(\tilde{x}(x,y), \tilde{y}(x,y))$$

Examples: many amusing warps of images

$$\tilde{x} = 2x$$
$$\tilde{y} = y$$

Image processing

An image processing operation typically defines a new image $g$ in terms of an existing image $f$.

The simplest operations are those that transform each pixel in isolation. These pixel-to-pixel operations can be written:

$$g(x,y) = t(f(x,y))$$

Examples: threshold, RGB $\rightarrow$ grayscale

Note: a typical choice for mapping to grayscale is to apply the YIQ television matrix and keep the $Y$.

$$\begin{bmatrix}
    Y \\
    I \\
    Q
\end{bmatrix} =
\begin{bmatrix}
    0.299 & 0.587 & 0.114 \\
    0.599 & -0.275 & -0.321 \\
    0.212 & -0.523 & 0.311
\end{bmatrix}
\begin{bmatrix}
    R \\
    G \\
    B
\end{bmatrix}$$

Noise

Image processing is also useful for noise reduction and edge enhancement. We will focus on these applications for the remainder of the lecture.

Common types of noise:

- Salt and pepper noise: contains random occurrences of black and white pixels
- Impulse noise: contains random occurrences of white pixels
- Gaussian noise: variations in intensity drawn from a Gaussian normal distribution
Ideal noise reduction

Practical noise reduction

How can we “smooth” away noise in a single image?

Convolution

One of the most common methods for filtering an image is called **convolution**.

In 1D, convolution is defined as:

\[
g(x) = \int_{-\infty}^{\infty} f(x') h(x-x') dx'
\]

where \( h(x) = h(-x) \).

Example:
**Discrete convolution**

For a digital signal, we define discrete convolution as:

\[ g[i] = f[i] \circ h[i] = \sum_{i'} f[i'] h[i - i'] \]

\[ = \sum_{i'} f[i'] \delta[i' - i] \]

where \( \delta[i] = \delta[-i] \).

**Aside:**

One can show that convolution has some convenient properties. Given functions \( a \) and \( c \):

\[ a \ast b = b \ast a \]

\[ (a \ast b) \ast c = a \ast (b \ast c) \]

\[ a \ast (b + c) = a \ast b + a \ast c \]

We'll make use of these properties later.

---

**Discrete convolution in 2D**

Similarly, discrete convolution in 2D becomes:

\[ g[i, j] = f[i, j] \circ h[i, j] = \sum_{i', j'} h[i', j'] h[i - i', j - j'] \]

\[ = \sum_{i', j'} h[i', j'] \delta[i' - i, j' - j] \]

where \( \delta[i, j] = \delta[-i, -j] \).

---

**Convolution in 2D**

In two dimensions, convolution becomes:

\[ g(x, y) = f(x, y) \ast h(x, y) \]

\[ = \int \int f(x', y') h(x' - x, y' - y) dx' dy' \]

\[ = \int \int f(x', y') h(x' - x, y' - y) dx' dy' \]

where \( h(x, y) = h(-x, -y) \).

---

**Convolution representation**

Since \( f \) and \( h \) are defined over finite regions, we can write them out in two-dimensional arrays:

<table>
<thead>
<tr>
<th></th>
<th>128</th>
<th>54</th>
<th>9</th>
<th>78</th>
<th>100</th>
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<tbody>
<tr>
<td>145</td>
<td>98</td>
<td>240</td>
<td>233</td>
<td>86</td>
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<td>89</td>
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<td>228</td>
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<td>90</td>
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<td>128</td>
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<td></td>
</tr>
<tr>
<td>221</td>
<td>154</td>
<td>97</td>
<td>123</td>
<td>0</td>
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</tr>
</tbody>
</table>

**Note:** This is not matrix multiplication!

Q: What happens at the edges?
Mean filters

How can we represent our noise-reducing averaging filter as a convolution diagram (known as a mean filter)?

\[ \begin{bmatrix} \frac{1}{m^2} \end{bmatrix} \]

Gaussian filters

Gaussian filters weight pixels based on their distance from the center of the convolution filter. In particular:

\[ h(x, y) = \frac{e^{-\left(x^2 + y^2\right)/(2\sigma^2)}}{C} \]

This does a decent job of blurring noise while preserving features of the image.

What parameter controls the width of the Gaussian?

What happens to the image as the Gaussian filter kernel gets wider?

What is the constant C? What should we set it to?

\[ C = \sum e^{\left|x^2 + y^2\right|/\sigma^2} \]

Effect of mean filters

Effect of Gaussian filters
Median filters

A median filter operates over a region by selecting the median intensity in the region.

What advantage does a median filter have over a mean filter? (acts like an outlier rejection filter)

Is a median filter a kind of convolution?

Effect of median filters

Gaussian noise  Salt and pepper noise

3x3

5x5

7x7

Comparison: Gaussian noise

<table>
<thead>
<tr>
<th>Mean</th>
<th>Gaussian</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>3x3</td>
<td></td>
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<tr>
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<tr>
<td>7x7</td>
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</tbody>
</table>

Comparison: salt and pepper noise

<table>
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**Edge detection**

One of the most important uses of image processing is **edge detection**:
- Really easy for humans
- Really difficult for computers
- Fundamental in computer vision
- Important in many graphics applications

---

**What is an edge?**

- Step
- Ramp
- Line
- Roof

Q: How might you detect an edge in 1D?

\[
\frac{df}{dx} \text{ thrsh.} \Rightarrow \text{edge}
\]

\[
g = h * f
\]

\[
\hat{h} = [0 -1 \ 1 0]
\]

---

**Gradients**

The gradient is the 2D equivalent of the derivative:

\[
\nabla f(x,y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)
\]

Properties of the gradient:
- It's a vector
- Points in the direction of maximum increase of \(f\)
- Magnitude is rate of increase

How can we approximate the gradient in a discrete image?

\[
\begin{align*}
\hat{g}_x[i,j] &= g[i+1,j] - g[i,j] \\
\hat{g}_y[i,j] &= g[i,j+1] - g[i,j]
\end{align*}
\]

\[
\hat{h}_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \hat{h}_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

---

**Less than ideal edges**

Pixels plotted
Steps in edge detection

Edge detection algorithms typically proceed in three or four steps:

- **Filtering**: cut down on noise
- **Enhancement**: amplify the difference between edges and non-edges
- **Detection**: use a threshold operation
- **Localization**: optional; estimate geometry of edges, which generally pass between pixels

Edge enhancement

A popular gradient magnitude computation is the Sobel operator:

$$s_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$s_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

(pro-flipped)

We can then compute the magnitude of the vector $(s_x, s_y)$.

Results of Sobel edge detection

![Original image](image1)

![Smoothed image](image2)

![Sx + 128](image3)

![Sy + 128](image4)

![Magnitude image](image5)

![Threshold = 64](image6)

![Threshold = 128](image7)

Second derivative operators

The Sobel operator can produce thick edges. Ideally, we're looking for infinitely thin boundaries.

An alternative approach is to look for local extrema in the first derivative plus where the change in the gradient is highest.

Q: A peak in the first derivative corresponds to what in the second derivative?

Q: How might we write this as a convolution filter?
Localization with the Laplacian

An equivalent measure of the second derivative in 2D is the Laplacian:

$$\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Using the same arguments we used to compute the gradient filters, we can derive a Laplacian filter to be:

$$\Delta = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta f = h_x \ast f + h_y \ast f = (h_x \ast f + h_y \ast f)$$

(The symbol $\Delta$ is often used to refer to the discrete Laplacian filter.)

Zero crossings of this filter correspond to positions of zero gradient. These zero crossings can be used to localize edges.

Marching squares

We can convert these signed values into edge contours using a "marching squares" technique:

$$s = f - \Delta f$$

$$= [I] \ast f \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= (I - \Delta) \ast f$$

Sharpening with the Laplacian

$$\text{Sharpen} = (I - \lambda \Delta)$$

$$\lambda \Delta = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Why does the sign make a difference?

How can you write each filter that makes each bottom image?
Summary

What you should take away from this lecture:

• The meanings of all the boldfaced terms.
• How noise reduction is done
• How discrete convolution filtering works
• The effect of mean, Gaussian, and median filters
• What an image gradient is and how it can be computed
• How edge detection is done
• What the Laplacian image is and how it is used in either edge detection or image sharpening