Dot Product

The dot product or inner product of two vectors is a very useful operation in computer graphics and is applied in numerous ways.

These notes are a short review of what the dot product is and some examples of how it gets used.

Definition

\[ v \cdot w = v_1w_1 + v_2w_2 + v_3w_3 \]
\[ = \|v\|\|w\|\cos(\theta) \]

if \( v \) is a unit vector, then
\[ v \cdot w = v_1w_1 + v_2w_2 + v_3w_3 \]
\[ = \|v\|\cos(\theta) \]

and so \( v \cdot w \) is the length of the projection of \( w \) onto \( v \).

Reference

Section A.3, Dot Products and Distances, *Computer Graphics, Principles and Practice*, Foley, van Dam
Illustration of $V \cdot W$

$V \cdot W = V_x W_x + V_y W_y$

$= \|V\| \|W\| \cos(\alpha) + \|V\| \|W\| \sin(\alpha)$

$= \|V\| \|W\| \left[ \cos(\alpha_x) + \sin(\alpha_x) \sin(\alpha_y) \right]$

$= \|V\| \|W\| \cos(\alpha_x - \alpha_y)$

$= \|V\| \|W\| \cos(\theta)$

The cosine is a useful function ...

if both $v$ and $w$ are unit vectors, then

$v \cdot w = \sum v_i w_i + v_3 w_3

= \|v\| \|w\| \cos(\theta)

= \cos(\theta)

and so $v \cdot w$ is just the cosine of the angle between the vectors

Unit vectors

The dot product of $v$ with itself is

$v \cdot v = v_1^2 + v_2^2 + v_3^2

= \|v\|^2 \cos(0)

= \|v\|^2

and so $v \cdot v$ is the square of its length

and if $v$ is a unit vector then $v \cdot v$ is 1

the columns of a rotation matrix are perpendicular unit vectors

$a \cdot a = \cos \theta \cos \theta + \sin \theta \sin \theta = 1$

$a \cdot b = \cos \theta (-\sin \theta) + \sin \theta \cos \theta = 0$

and so the transpose of a rotation matrix is its inverse

$$
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

Front facing polygon?

note that the transpose is rotation through $-\theta$, since

$-\sin(-\theta) = \sin(\theta)$, which also shows that the transpose is the inverse of the original rotation matrix.
Equation of a line

\[ Ax + By + C = 0 \]

All vectors \((x, y)\) for which \((A, B) \cdot (x, y) = -C\)

Where on ray is closest approach to \(C\)?

\[ RC = C - R_o \]
\[ R_{\text{dir}} \text{ is ray origin} \]
\[ R_{\text{dir}} \text{ is ray direction vector (unit vector)} \]
so
\[ t_{\alpha} = RC \cdot R_{\text{dir}} \]

Diffuse reflection of light

\[ \frac{A_p}{A} = \cos \theta \]
\[ N \cdot L = \|N\| \cos \theta \]
\[ = \|N\| \frac{A_p}{A} \]
so \(N \cdot L\) gives a scaled value for diffuse reflected light intensity